PROBLEM SET 3

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1. 25 Points

CONSIDER A DIODE OF VOLTAGE V_0 AND gap length d.
Let a current density J be composed of two species
such that $J_1 = \alpha J$ and $J_2 = (1-\alpha) J$ (so that $J = J_1 + J_2$).

Let the mass of ions in species 1 be m_1 and those of species 2 be m_2. What is the effective mass m_{eff} that should be used in the resulting Child Langmuir Law:

$$J = \frac{4}{9} \varepsilon_0 \left(\frac{2q}{m_{eff}}\right)^{1/2} \frac{V_0^{3/2}}{d^2}$$

(Both ion species have charge q).

2. 25 Points

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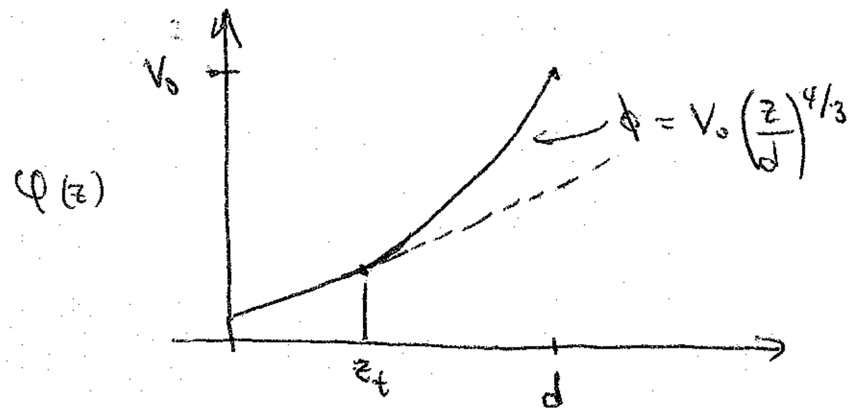
PROBLEM 2 CONSIDER THE FOLLOWING DIODE OF voltage V_0 AND LENGTH d.

SUPPOSE AT SOME TIME $t_p > \tau \equiv \frac{3d}{\left(\frac{2qV_0}{m}\right)^{1/2}}$ THE CURRENT IS ABRUPTLY TURNED OFF. WHAT VOLTAGE WAVEFORM IS REQUIRED TO ENSURE THAT THE ELECTRIC FIELD AT THE TAIL OF THE PULSE IS IDENTICAL TO THE CHILD-LANGMUIR ELECTRIC FIELD?



Problem 3 25 Points S. M. Lund

TED Problem 1

1/ Consider a $\perp$ unbunched ion beam described by

$$f_\perp(\vec{x}_\perp, \vec{x}_\perp', s) \quad \sim \quad \text{single particle distribution. Satisfying Vlasov's equation.}$$

$$H_\perp = \frac{1}{2}\vec{x}_\perp'^2 + \frac{\kappa_x(s)}{2}x^2 + \frac{\kappa_y(s)}{2}y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

$$\nabla_\perp^2\phi = -\frac{q}{\varepsilon_0}\int d^2x'\, f(\vec{x}_\perp, \vec{x}_\perp', s)$$

$$\phi(r = r_p) = 0$$

Grounded pipe boundary condition. r_p = pipe radius.

a) What are the first-order particle equations of motion for $d/ds\, \vec{x}_\perp$ and $\frac{d}{ds}\vec{x}_\perp'$ derived from $H_\perp$?

b) Using the results of part a), what is the 2nd-order particle equation of motion for $\frac{d^2}{ds^2}\vec{x}_\perp$?

c) Use the particle equations of motion to calculate d/ds of the single-particle Hamiltonian $H_\perp$ and the "angular momentum"

$$P_\theta \equiv xy' - yx'.$$

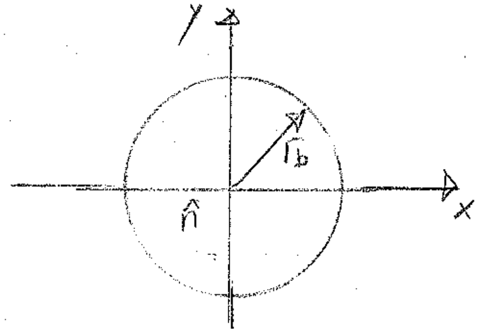
I.e., $\frac{d}{ds}H_\perp = ?$, $\frac{d}{ds}P_\theta = ?$

d) Use the expressions of part c) to show that for $\kappa_x = \text{const}$, $\kappa_y = \text{const}$, and $f_\perp = f_\perp(H_\perp)$ that $H_\perp = \text{const}$. Here $f(H_\perp)$ can be any function of $H_\perp$ with $f(H_\perp) \geq 0$.

e) Use the expressions of part c) to show that for axisymmetric beams ($\partial/\partial\theta = 0$; θ = azimuthal angle) with $\kappa_x = \kappa_y = \kappa(s)$ and $f_\perp = f_\perp(H_\perp)$ that $P_\theta = \text{const}$.

TED Problem 2

2/ Consider a uniform density beam in free-space with circular cross-section, edge radius r_b, and uniform in z $(\partial/\partial z = 0)$.



r_b = beam edge radius.
$r = \sqrt{x^2+y^2}$
$\hat{n}$ = const.
$\lambda = q\hat{n}\pi r_b^2$ = line-charge

a) <u>Directly</u> construct the solution to Poisson's equation

$$\left(\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y}\right)\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = \frac{-q}{\varepsilon_0}\begin{cases}\hat{n}, & r < r_b \\ 0, & r > r_b.\end{cases}$$

satisfying

$$\frac{\partial\phi}{\partial\theta} = 0 \quad \text{and} \quad \lim_{r\to\infty} -\frac{\partial\phi}{\partial r} = \frac{\lambda}{2\pi\varepsilon_0 r}$$

b) Take derivatives of the interior solution $(r < r_b)$ in part a) to obtain formulas for

$$E_x = -\frac{\partial\phi}{\partial x}$$

$$E_y = -\frac{\partial\phi}{\partial y}$$

c) Show that the ellipsoidal beam formulas

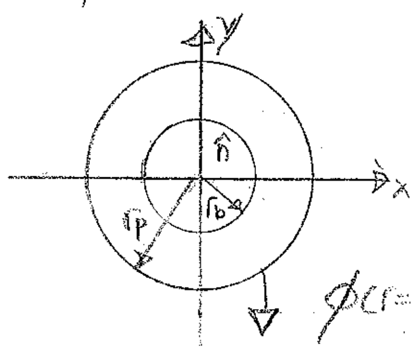
$$E_x = -\frac{\partial\phi}{\partial x} = \frac{\lambda}{\pi\varepsilon_0}\frac{x/r_x}{r_x+r_y}$$

$$E_y = -\frac{\partial\phi}{\partial y} = \frac{\lambda}{\pi\varepsilon_0}\frac{y/r_y}{r_x+r_y}$$

reduce to the results in part c) for a round beam with $r_x = r_y = r_b$.

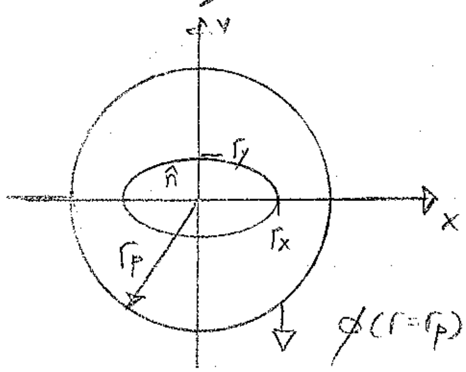
TED Problem 2

d) Would a grounded, conducting pipe of radius $r = r_p > r_b$ change the answers in part b)?



$r_p > r_b$

e) Would a grounded conducting pipe of radius $r = r_p > r_x, r_y$ change the fields calculated in class for the elliptical beam case with $r_x \neq r_y$? (no need to calculate any changes, just explain answer)



Problem 5 15 Points S.M. Lund P3/

TED Problem 3.

3/ For a KV distribution:

$$n(x,y) = \int dx' dy' f_\perp = \begin{cases} \hat{n}, & \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1 \\ 0; & \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1 \end{cases}$$

Use this result to verify the formulas

$$r_x = 2\langle x^2 \rangle_\perp^{1/2}$$
$$r_y = 2\langle y^2 \rangle_\perp^{1/2}$$

Hint: Integrals may be more easily carried out if the elliptical integration domain is transformed to a circular domain.

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} = 1 \qquad \text{Elliptical beam edge}$$

$$x = r_x \rho \cos\psi$$
$$y = r_y \rho \sin\psi \quad \longrightarrow \quad \rho^2\cos^2\psi + \rho^2 \sin^2\psi = 1$$
$$\rho^2 = 1 \quad \text{beam edge.}$$

can carry out integration in $\rho - \psi$ variables to simplify.