Problem #1
10 points

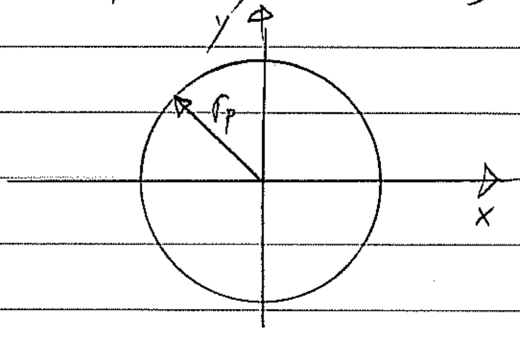
Problem Set #11
NE290H Barnard and Lund
Due April 22, 2009

S.M. Lund P2/

TCE Problem 2

2/ Image Charges on a Cylindrical Pipe.

Consider a perfectly conducting pipe of radius r_p:



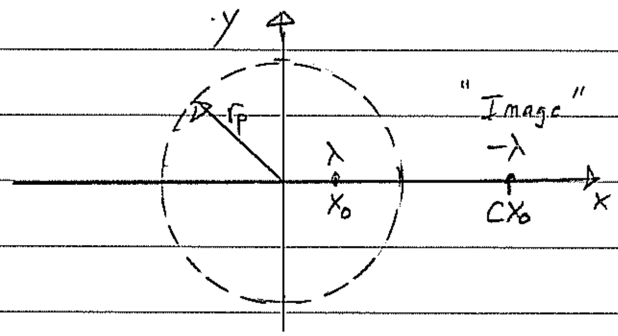
$\frac{\partial}{\partial z} = 0$

$\phi(r = r_p) = \text{const}$

A/ Show that the formula for a line-charge λ at the origin in free-space is:

$$\phi(r) = \frac{-\lambda}{2\pi\varepsilon_0} \ln r + \text{const}$$

B/ Use the formula in part A/ to show that a solution to the interior problem $|\vec{x}_\perp| < r_p$ can be found for a line charge λ at coordinate $x = x_0$ by superimposing the direct charge and an image charge at $x = C x_0$. Calculate C for cylindrical geometry.



Images can be superimposed to obtain the Green's Function for the 2D calculation of ϕ within the cylinder.

TCE Problem 4

Problem #2
10 points

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4/ Image charge field of a centered elliptical beam in a round pipe.

Take $X = Y = 0$ and calculate the image field terms for a uniform density elliptical beam:

$$\rho(x,y) = \begin{cases} \frac{\lambda}{\pi r_x r_y} ; & \left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 < 1 \\ 0 ; & \text{otherwise} \end{cases}$$

in a cylindrical pipe of radius $r_p > r_x, r_y$.

In class it was shown that the image field can be expanded as:

$$\underset{\sim}{E}^i = E_y^i + i E_x^i = \sum_{n=1}^{\infty} \underset{\sim}{C}_n (x+iy)^{n-1} \qquad i = \sqrt{-1}$$

$$\underset{\sim}{C}_n = \frac{-1}{2\pi i \varepsilon_0} \int d^2x \; \rho(x,y) \cdot \frac{(x-iy)^n}{r_p^{2n}}$$

Calculate the first nonvanishing term for $r_x \neq r_y$, for

E_x^i

E_y^i

compare these answers to the results presented in class that were derived by Lee et. al for an elliptical beam displaced along the x-axis.

TCE Problem 3

Problem #3
20 points

S.M. Lund P3/

3/ Axisymmetric Envelope Equation

Take

$$X = 0 = Y \quad \text{Zero centroid} \quad ; \quad \varepsilon_x = \varepsilon_y = \varepsilon \quad \text{equal emittances}$$

$$r_x = r_y \equiv r_b \quad \text{Round beam}$$

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const} \quad \text{Cont. Focusing}$$

and a uniform density beam of circular cross-section in a cylindrical pipe of radius $r_p > r_b$.

A/ Calculate $\frac{\partial \phi}{\partial x}$ inside the beam and show that the x-particle equation of motion is:

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + k_{\beta 0}^2 x - \frac{Q}{r_b^2} x = 0$$

$$Q \equiv \frac{q\lambda}{2\pi\varepsilon_0 m \gamma_b^3 \beta_b^2 c^2} \quad ; \quad \lambda \equiv q \hat{n} \pi r_b^2 = \text{const.}$$

B/ Parallel steps in class to derive the envelope equation:

$$r_b'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} r_b' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

where

$$\varepsilon_x \equiv 4 \left[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle x x' \rangle_\perp^2 \right]^{1/2}$$

C/ For a non-uniform density ~~axisymmetric beam with $\phi = \phi(r)$~~ the particle equation of motion in A/ becomes:

$$x'' + \frac{(\gamma_b \beta_b)'}{(\gamma_b \beta_b)} x' + k_{\beta 0}^2 x = -\frac{q}{m \gamma_b^3 \beta_b^2 c^2} \frac{\partial \phi}{\partial x}$$

show that the envelope equation is now:

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$$r_b'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)'} r_b' + k_{\beta 0}^2 r_b + \frac{4q \left\langle x \frac{\partial\phi}{\partial x} \right\rangle_\perp}{m\gamma_b^3\beta_b^2 c^2 r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0$$

where

$$r_b \equiv 2\langle x^2\rangle^{1/2} \qquad \langle x^2\rangle = \frac{\int_0^{r_b} dr\, r^3 \rho(r)}{\int_0^{r_b} dr\, r\, \rho(r)}$$

$\rho(r)$ = beam charge density.

In earlier problem sets you showed that:

$$\left\langle x \frac{\partial\phi}{\partial x} \right\rangle_\perp = \frac{-\lambda}{8\pi\varepsilon_0} \qquad \lambda \equiv 2\pi\int_0^{r_b} dr\, r\, \rho(r) = \text{const.}$$

So this results in the same statistical envelope equation as in part B/ with Q defined by λ.

D/ Take: $\gamma_b\beta_b = \text{const}$ and

$$r_b(s) = \underbrace{r_{b0}}_{\text{const.}} + \underbrace{\delta r_b}_{\text{const.}} e^{iks} \qquad |\delta r_b| \ll r_{b0}$$

$k = \text{const.}$

and require that r_{b0} satisfy the envelope equation with $\delta r_b = 0$. Then require that the form above satisfy the envelope equation to linear order in δr_b. Show that for nontrivial solutions

$$k^2 = 2k_{\beta 0}^2 + 2k_\beta^2$$

where

$$k_\beta^2 \equiv k_{\beta 0}^2 - \frac{Q}{r_{b0}^2} \equiv \text{depressed } \beta\text{-tron wavenumber}$$

–or–

$$k = k_{\beta 0}\sqrt{2 + 2(\sigma/\sigma_0)^2} \qquad \frac{\sigma}{\sigma_0} = \frac{k_\beta}{k_{\beta 0}}$$

Problem # 4
15 points

S. M. Lund P7/

TCE Problem 7

7/ Problem - Normalized Emittance Conservation

The rms measures of the "normalized" beam emittance are

$$\varepsilon_{nx} = 4\gamma_b\beta_b\left[\langle x^2\rangle_\perp\langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right]^{1/2} \equiv \gamma_b\beta_b\,\varepsilon_x$$
$$\varepsilon_{ny} = 4\gamma_b\beta_b\left[\langle y^2\rangle_\perp\langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2\right]^{1/2} \equiv \gamma_b\beta_b\,\varepsilon_y$$

where ε_x and ε_y are known as regular, unnormalized, emittances.

$$\varepsilon_x \equiv 4\left[\langle x^2\rangle_\perp\langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right]^{1/2}$$
$$\varepsilon_y \equiv 4\left[\langle y^2\rangle_\perp\langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2\right]^{1/2}$$

A/ For a uniform density, elliptical beam with envelope radii

$$r_x = 2\langle x^2\rangle_\perp^{1/2}$$
$$r_y = 2\langle y^2\rangle_\perp^{1/2}$$

a particle moving within the beam has equations of motion

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x - \frac{2Q\,x}{(r_x+r_y)r_x} = 0$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y - \frac{2Q\,y}{(r_x+r_y)r_y} = 0$$

Show that subject to these equations, that ε_{nx} is a constant of the motion, i.e.,

$$\varepsilon_{nx} = \text{const.}$$

Hint: It is easier to do this directly from the eqns of motion than via transforms.

Does the same result hold for ε_{ny}?

B/ If the equations of motion are generalized to contain terms with nonlinear applied focusing forces, i.e.,

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x - \frac{2Q\,x}{(r_x+r_y)r_x} = F_x(x,y)$$

where $F_x(x,y)$ has nonlinear terms (i.e., $F_x(x,y) = Cx^2,\ Dxy,\ Ex^2y$, ... etc. with C, D, E constants) derive an equation for $\frac{d}{ds}\varepsilon_{nx}^2$. Do you expect a solution with $\varepsilon_{nx}^2 = \text{const}$? Why (qualitative only)?