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USPAS
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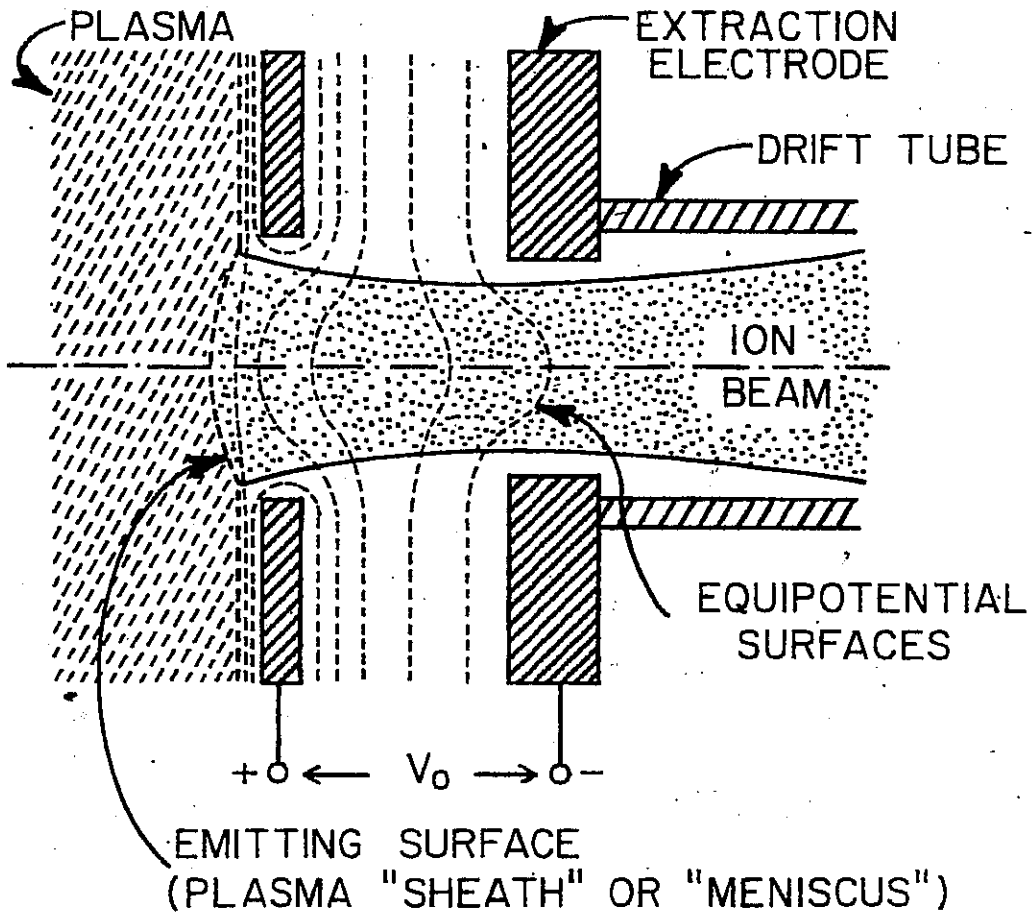
Injectors and longitudinal physics -- I

1. Child-Langmuir Law
   (Reiser 2.5.2, Appendix 1)
2. Pierce electrodes
3. Transients in injectors
4. Injector choices

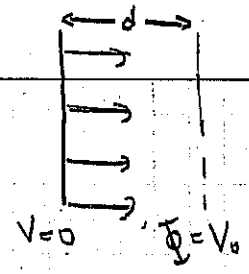


OR "HOT PLATE"

REISER, FIGURE 1.2

- DOPED TUNGSTEN
- ALUMINO-SILICATE

I CHILD-LANGMUIR EMISSION



ASSUME EMISSION IS PLANAR 1-D:

$$J = \rho v_z \qquad (1)$$

$$\frac{1}{2} m v_z^2 = -q\Phi \qquad (2) \Rightarrow v_z = \left(\frac{-2q\Phi}{m}\right)^{1/2}$$

$$\frac{d^2\Phi}{dz^2} = \frac{\rho}{\varepsilon_0} \qquad (3)$$

(NOTE $\Phi = -\phi$ ← ACTUAL e.s. potential)

CONTINUITY EQUATION (1D) $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}\rho v_z = 0$

for time steady emission ρv_z = constant = J

$$\Rightarrow \frac{d^2\Phi}{dz^2} = \frac{J}{\varepsilon_0 v_z} = \frac{J}{\varepsilon_0\left(\frac{-2q\Phi}{m}\right)^{1/2}}$$

MULTIPLYING BY $\frac{d\Phi}{dz}$ AND INTEGRATING:

$$\frac{\Phi'^2}{2} = \frac{J m^{1/2}}{\varepsilon_0 (2q)^{1/2}} 2\Phi^{1/2} + \text{const}$$

Assume $\Phi' = 0$ at $z=0$ (Space-charge limited emission)

$\Phi = 0$ at $z=0 \Rightarrow$ const $= 0$

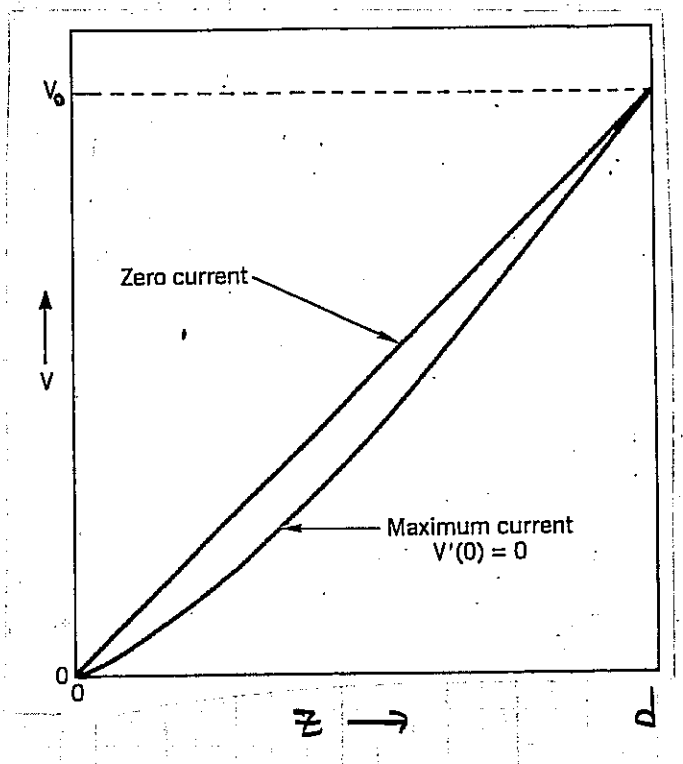
$$\frac{d\Phi}{dz} = \left(\frac{4J}{\varepsilon_0}\right)^{1/2}\left(\frac{m\Phi}{2q}\right)^{1/4}$$

$$\Rightarrow \frac{4}{3}\Phi^{3/4} = \left(\frac{4J}{\varepsilon_0}\right)^{1/2}\left(\frac{m}{2q}\right)^{1/4} z \Rightarrow \Phi(x) = \left(\frac{3}{4}\right)^{4/3}\left(\frac{4J}{\varepsilon_0}\right)^{2/3}\left(\frac{m}{2q}\right)^{1/3} z^{4/3}$$

If $\Phi = V_0$ at $z = d$ $\Rightarrow$ $\Phi = V_0 \left(\frac{z}{d}\right)^{4/3}$

$$\Rightarrow \quad V_0 = \left(\frac{3}{4}\right)^{4/3} \left(\frac{4J}{\varepsilon_0}\right)^{2/3} \left(\frac{m}{2q}\right)^{1/3} d^{4/3}$$

$$\text{or} \quad J = \frac{4}{9}\varepsilon_0 \left(\frac{2q}{m}\right)^{1/2} \frac{V_0^{3/2}}{d^2}$$



NOTE THAT IF WE MULTIPLY J BY THE BEAM AREA πr_b^2, AND DIVIDE BY $v = \left(\frac{2qV_0}{m}\right)^{1/2}$

$$\Rightarrow \lambda = \frac{4\pi\varepsilon_0 V}{9}\left(\frac{r_b^2}{d^2}\right) \qquad \text{RECALL } Q \equiv \frac{\lambda}{4\pi\varepsilon_0 V} \text{ (NON-RELAT.)}$$

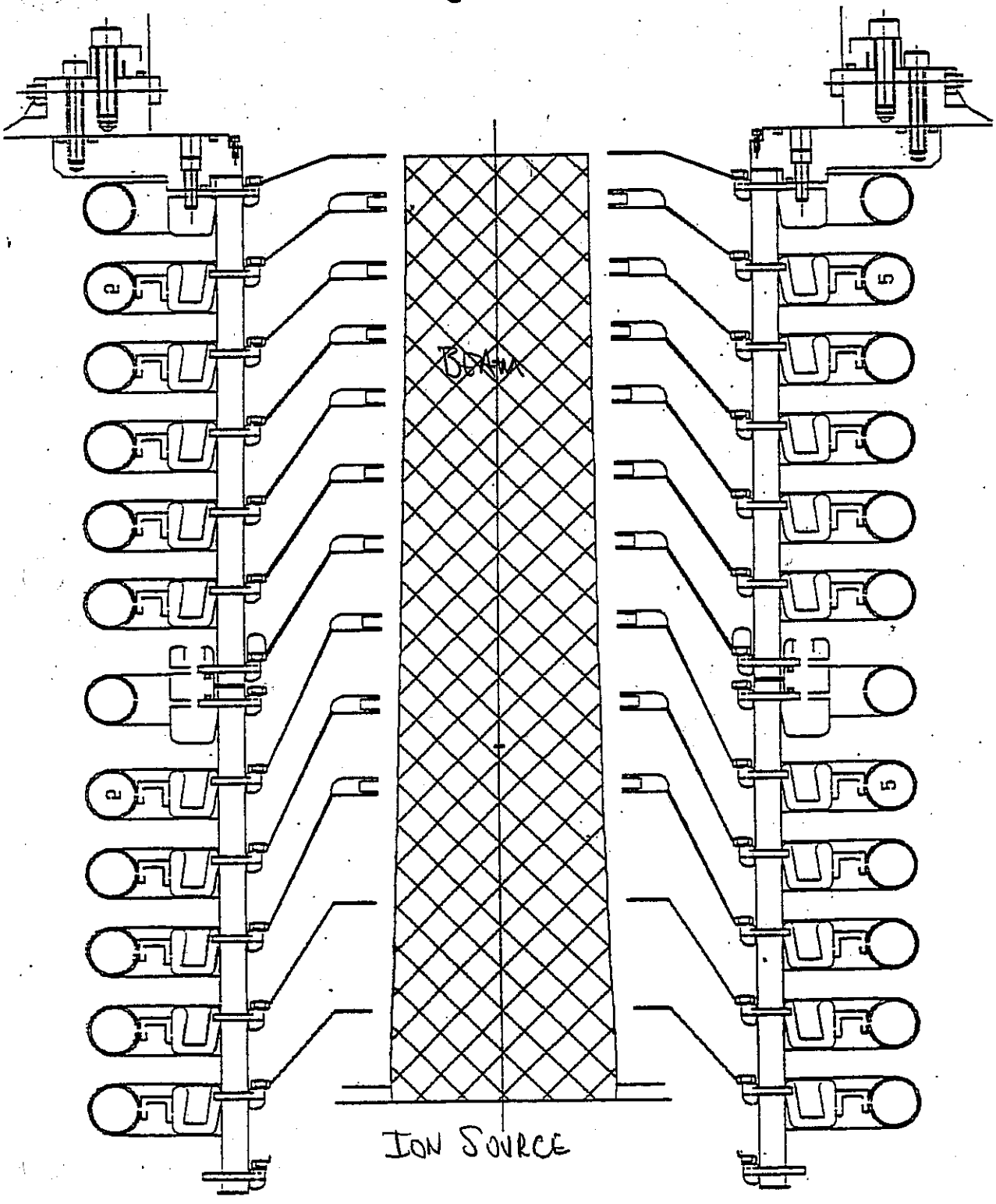
$$\Rightarrow Q = \frac{1}{9}\left(\frac{r_b^2}{d^2}\right) \quad \text{or as a function of } z: \; Q(z) = \frac{1}{9}\left(\frac{r_b^2}{z^2}\right)$$

PERVEANCE

PIERCE COLUMN

$V \sim z^{4/3}$

$E \sim z^{1/3}$



[redacted], WE DERIVED THE PARAXIAL RAY EQUATION FOR PARTICLES IN AXISYMMETRIC SYSTEMS:

$$\underbrace{r'' + \frac{\gamma'}{\beta^2\gamma} r'}_{\text{INERTIAL}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma} r}_{E_r} + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r}_{V_\theta B_z \,-\, \text{CENTRIFUGAL}} - \underbrace{\left(\frac{p_\theta}{\gamma\beta m c}\right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}}_{\text{SELF-FIELD}} = 1$$

$$\theta' = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{\omega_c}{2\gamma\beta c}$$ ← CONSTANCY & DEFINITION OF CANONICAL MOMENTUM

ENVELOPE EQUATION FOR AXISYMMETRIC BEAM

$$r_b'' + \frac{\gamma' r_b'}{\beta^2\gamma} + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b - \frac{4\langle p_\theta\rangle^2}{(\gamma m\beta c)^2 r_b^3} - \frac{\epsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0$$

$$\epsilon_r^2 \equiv 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle r r'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right)$$

RETURNING TO PARAXIAL ENVELOPE EQUATION:

(for $\beta << 1$) $$r_b'' + \frac{\beta'}{\beta} r_b' + \left[\frac{1}{2}\frac{\beta'^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta}\right] r_b - \frac{Q}{r_b} = 0$$

for $r_b'' = \frac{\beta'}{\beta} r_b' = 0$

If $Q = V_0 \left(\frac{z}{d}\right)^{4/3}$

$$v = C z^{2/3}$$

$$v' = \frac{2}{3} C z^{-1/3}$$

$$v'' = \frac{-2}{9} C z^{-4/3}$$

$$\Rightarrow \left[\frac{1}{2}\frac{\beta'^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta}\right] r_b^2 = Q$$

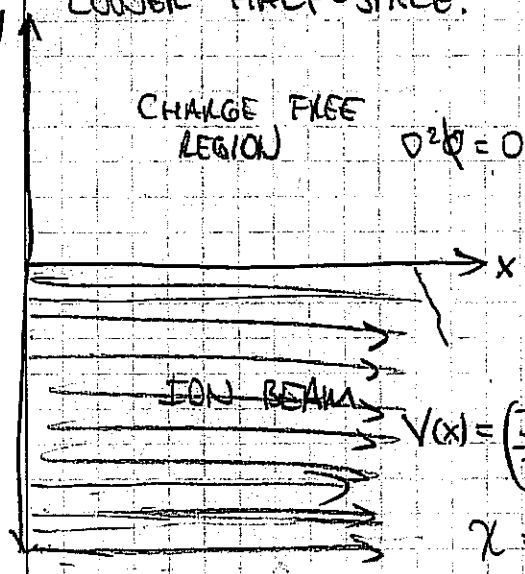
$$\left[\frac{2}{9}\frac{1}{z^2} \qquad -\frac{1}{9}\frac{1}{z^2}\right]$$

$$\Rightarrow Q(z) = \frac{1}{9}\frac{r_b^2}{z^2}$$

So Child-Langmuir flow satisfies the PARAXIAL ENVELOPE EQUATION FOR A CONSTANT BEAM RADIUS (AS IT SHOULD!)

PIERCE ELECTRODES: GOING BEYOND PARAXIAL APPROXIMATION

CONSIDER THE CASE A BEAM WHICH FILLS THE LOWER HALF-SPACE.



CHARGE FREE REGION $\nabla^2\phi = 0$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$V(x) = \left(\frac{J}{\chi}\right)^{2/3} x^{4/3}$$

$$\chi = \left(\frac{4\epsilon_0}{9}\right)\sqrt{\frac{2q}{m}}$$

FIND SOLUTION SUCH THAT

$$\frac{\partial\phi}{\partial y}(x, y=0) = 0$$

$$\& \quad \phi(x, y=0) = V(x)$$

PIERCE'S SOLUTION: LET THE POTENTIAL BE THE REAL PART OF

$$\phi + iW = V(x+iy) \equiv V(z) \qquad z \equiv x+iy$$

NOTE THAT FOR ANY $V(z)$ WITH derivatives that exist independent of direction (analytic) the real part of $V(z)$ satisfies Laplace's equation: $\frac{\partial^2 \mathrm{Re}[V]}{\partial x^2} + \frac{\partial^2 \mathrm{Re}[V]}{\partial y^2} = 0$.

$$\frac{\partial\phi}{\partial x} = \mathrm{Re}\left[\frac{dV}{dz}\right] \qquad \frac{\partial\phi}{\partial y} = \mathrm{Re}\left[i\frac{dV}{dz}\right]$$

$$\frac{\partial^2\phi}{\partial x^2} = \mathrm{Re}\left[\frac{d^2V}{dz^2}\right] \qquad \frac{\partial^2\phi}{\partial y^2} = -\mathrm{Re}\left[\frac{d^2V}{dz^2}\right] \Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \mathrm{Re}\left[\frac{d^2V}{dz^2}\right] - \mathrm{Re}\left[\frac{d^2V}{dz^2}\right] = 0$$

CHOOSE $V(z) = \left(\frac{J}{\chi}\right)^{2/3} (x+iy)^{4/3}$

By construction $\phi(x, y=0) = V(x)$ ✓

$$\phi = \mathrm{Re}\left[\left(\frac{J}{\chi}\right)^{2/3} (x+iy)^{4/3}\right]$$

Let $x+iy = r\exp[i\theta]$
$(x+iy)^{4/3} = r^{4/3}\exp\left[\frac{4i\theta}{3}\right]$

$$= \left(\frac{J}{\chi}\right)^{2/3} (x^2+y^2)^{2/3} \mathrm{Re}\left[\exp\left[i\,\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]\right]$$

$$\boxed{\phi(x,y) = \left(\frac{J}{\chi}\right)^{2/3} (x^2+y^2)^{2/3} \cos\left[\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]}$$

Note that $\phi(x,y) = \phi(x,-y) \Rightarrow \frac{\partial\phi}{\partial y}(x, y=0) = 0$

$\phi = 0$ EQUIPOTENTIAL:

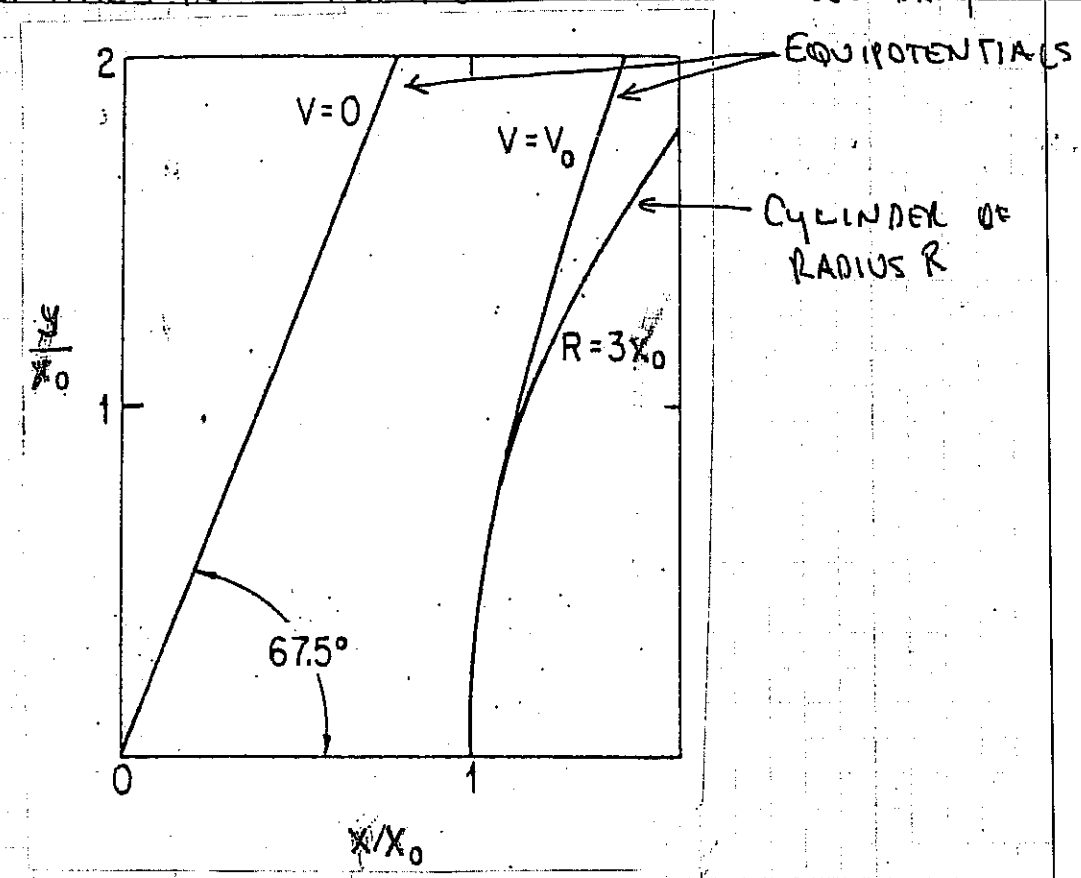
$$\Rightarrow 0 = \cos\left[\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]$$

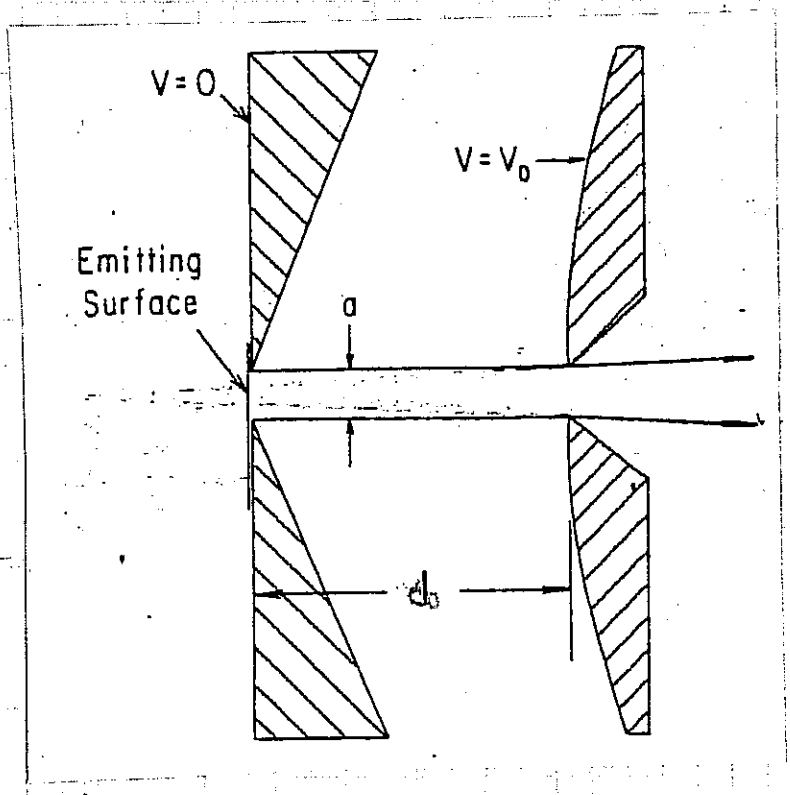
$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{3}{4}\left(\frac{\pi}{2}\right) = 67.5^\circ$$

FOR A GENERAL EQUIPOTENTIAL PASSING THROUGH X_0:

$$X_0^{4/3} = (x^2+y^2)^{2/3} \cos\left[\frac{4}{3}\tan^{-1}\left(\frac{y}{x}\right)\right]$$

EQUIPOTENTIALS FOR MATCHING TO A PLANAR BOUNDARY

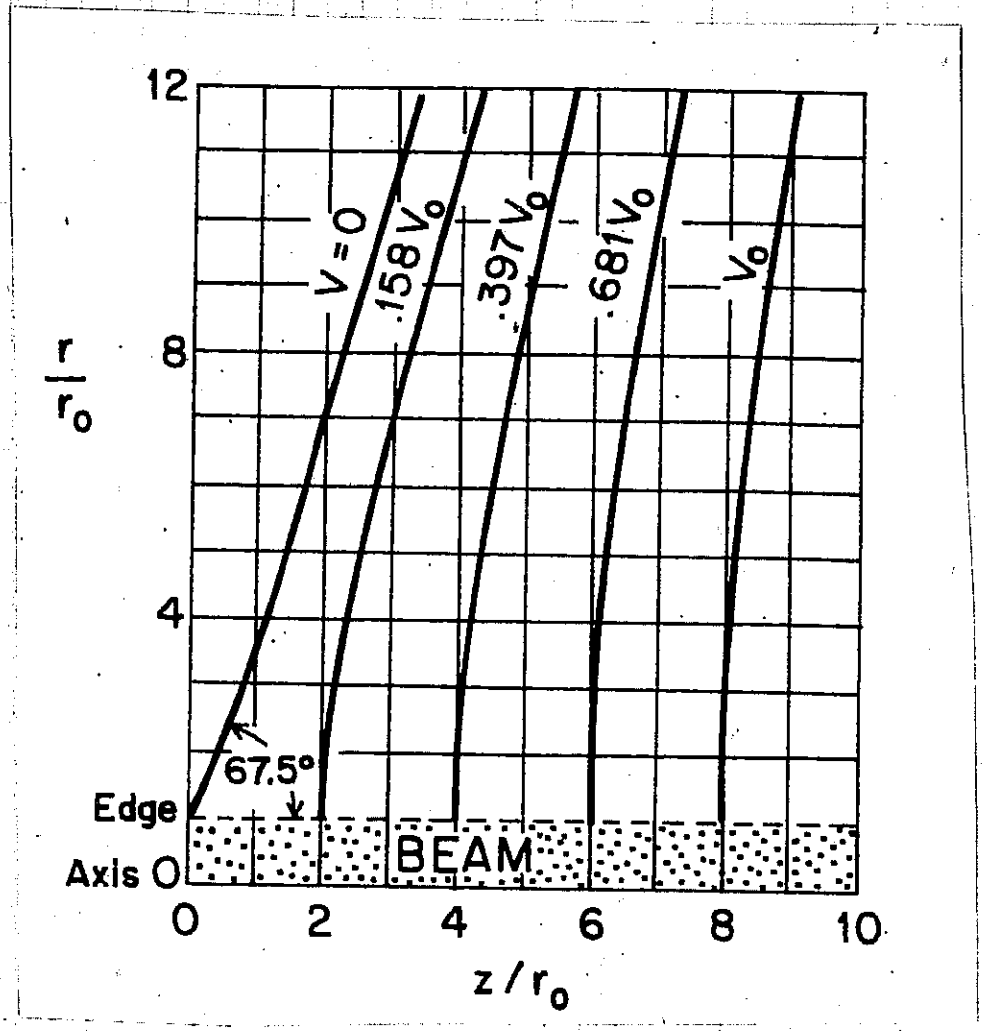




FIGURES FROM "LARGE ION BEAMS, FUNDAMENTALS OF GENERATION AND PROPAGATION" BY A.T. FORRESTER, Wiley, NY, 1988.

ELECTRODES FOR A LONG SLIT ION BEAM SYSTEM

PIERCE ELECTRODES FOR CIRCULAR BEAMS



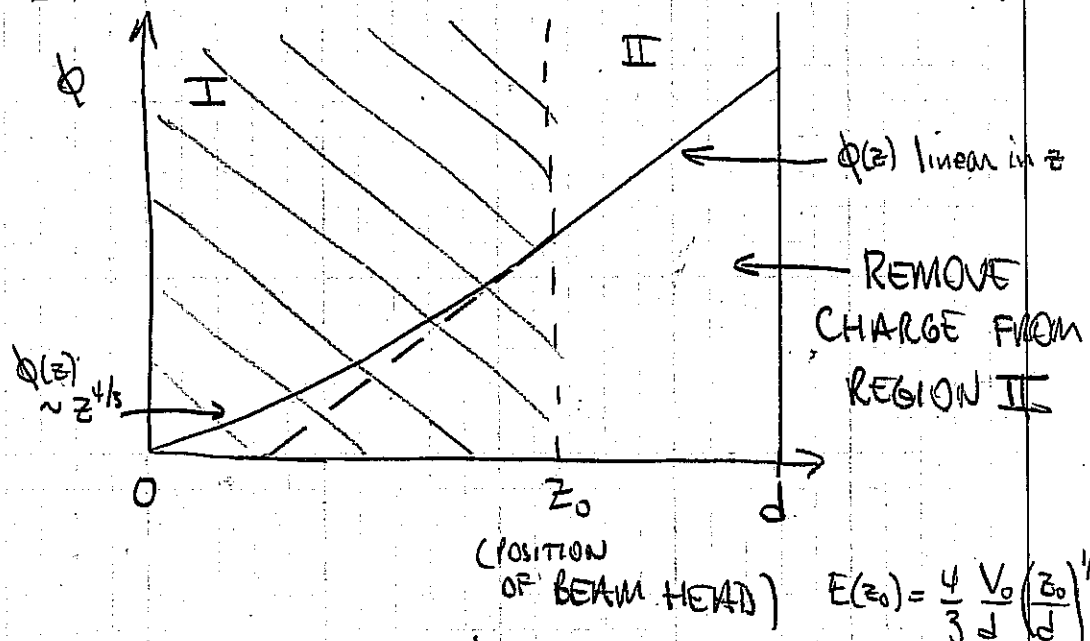
- SOLUTION is similar, but must be done numerically
- $\phi = 0$ is same as planar case

FIGURE FROM A.T. FORRESTER, "LARGE ION BEAMS, ..." Wiley, 1988

TRANSIENTS IN INJECTORS (LAMPEL & TIEFENBACK, Appl. Phys. Lett, 43, 57, 1983)

DURING TURN-ON THERE IS NO SPACE CHARGE IN FRONT OF BEAM, SO FIELDS MAY NOT BE GIVEN BY CHILD-LANGMUIR LAW. ⇒ CURRENT SPIKES POSSIBLE ⇒ ADVERSE TRANSVERSE COUPLING.

SOLUTION: ADJUST VOLTAGE ON DIODE SUCH THAT C-L LFIELD OCCURS EVERYWHERE THERE IS BEAM.



$$E(z_0) = \frac{4}{3}\frac{V_0}{d}\left(\frac{z_0}{d}\right)^{1/3}$$

$$\phi(d) = \int_0^{z_0} E(z)\,dz + \int_{z_0}^{d} E(z)\,dz$$

$$= V_0\left(\frac{z_0}{d}\right)^{4/3} + \frac{4}{3}\frac{V_0}{d}\left(\frac{z_0}{d}\right)^{1/3}(d - z_0)$$

$$= V_0\left[\frac{4}{3}\left(\frac{z_0}{d}\right)^{1/3} - \frac{1}{3}\left(\frac{z_0}{d}\right)^{4/3}\right]$$

(NOTE V_0 IS THE DESIRED STEADY STATE VOLTAGE ACROSS DIODE).

SO IF WE KNOW $z_0(t)$ WE CAN DETERMINE $\Phi(t)$.

$$\frac{1}{2} m \dot{z}_0^2 = q V_0 \left(\frac{z_0}{d}\right)^{4/3}$$

(SINCE BY CONSTRUCTION, HEAD OF BEAM TRAVELS AT CHILD-LANGMUIR VELOCITY LIKE ALL PARTICLES).

$$\dot{z}_0 = \left(\frac{2qV_0}{m}\right)^{1/2} \left(\frac{z_0}{d}\right)^{2/3}$$

$$\frac{dz_0}{z_0^{2/3}} = \left(\frac{2qV_0}{m}\right)^{1/2} \frac{dt}{d^{2/3}}$$

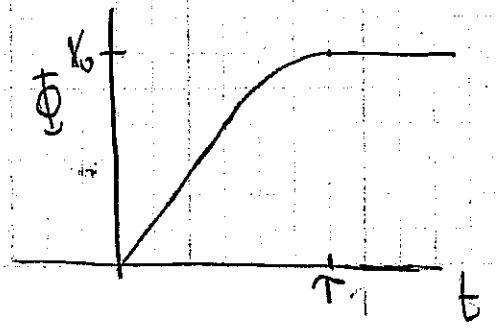
$$\Rightarrow 3 z_0^{1/3} = \left(\frac{2qV_0}{m}\right)^{1/2} \frac{t}{d^{2/3}} \quad \Rightarrow \quad t = \frac{3(z_0 d^2)^{1/3}}{\left(\frac{2qV_0}{m}\right)^{1/2}}$$

Let $\tau = \dfrac{3d}{\left(\frac{2qV_0}{m}\right)^{1/2}}$ = transit time across diode

$$\Rightarrow \frac{t}{\tau} = \left(\frac{z_0}{d}\right)^{1/3}$$

$$\Phi(d, z_0) = V_0 \left[\frac{4}{3}\left(\frac{z_0}{d}\right)^{1/3} - \frac{1}{3}\left(\frac{z_0}{d}\right)^{4/3}\right]$$

$$\Rightarrow \Phi(d, t) = \begin{cases} V_0 \left[\frac{4}{3}\left(\frac{t}{\tau}\right) - \frac{1}{3}\left(\frac{t}{\tau}\right)^4\right] & \text{for } 0 < t < \tau \\ V_0 & \text{for } t > \tau \end{cases}$$



INJECTOR CHOICES (cf. Kwan et al, NIMPRA, 464, 379 (2001).

CHILD-LANGMUIR $\Rightarrow J = \chi \dfrac{V^{3/2}}{d^2}$

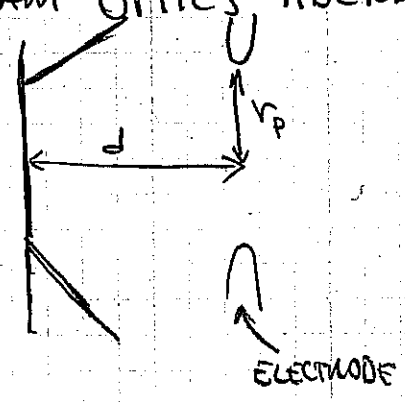
where $\chi = \frac{4}{9}\varepsilon_0 \left(\frac{2q}{m}\right)^{1/2}$

SOME CONSTRAINTS:

(1) VOLTAGE BREAKDOWN

EMPIRICALLY $V \lesssim 100\,\text{kV} \begin{cases} \left(\dfrac{d}{1\,\text{cm}}\right) & \text{for } d \le 1\,\text{cm} \\ \left(\dfrac{d}{1\,\text{cm}}\right)^{1/2} & \text{for } d \geq 1\,\text{cm} \end{cases}$

(2) BEAM OPTICS ABERRATIONS: $d \gtrsim 3 \text{ to } 4\, r_p$ (TYPICALLY)



NOTE THAT

$$J \sim \frac{V^{3/2}}{d^2} \sim \begin{cases} V^{-1/2} & \leftarrow d < 1\,\text{cm} \\ V^{-3/2} & \leftarrow d > 1\,\text{cm} \end{cases} \qquad I \sim \pi r_p^2 J \sim \left\{ V^{3/2} \right.$$

Thus current density decreases with size and voltage, but I increases.

Air
$V = 24s + 7\sqrt{s}$

from A. Faltens:

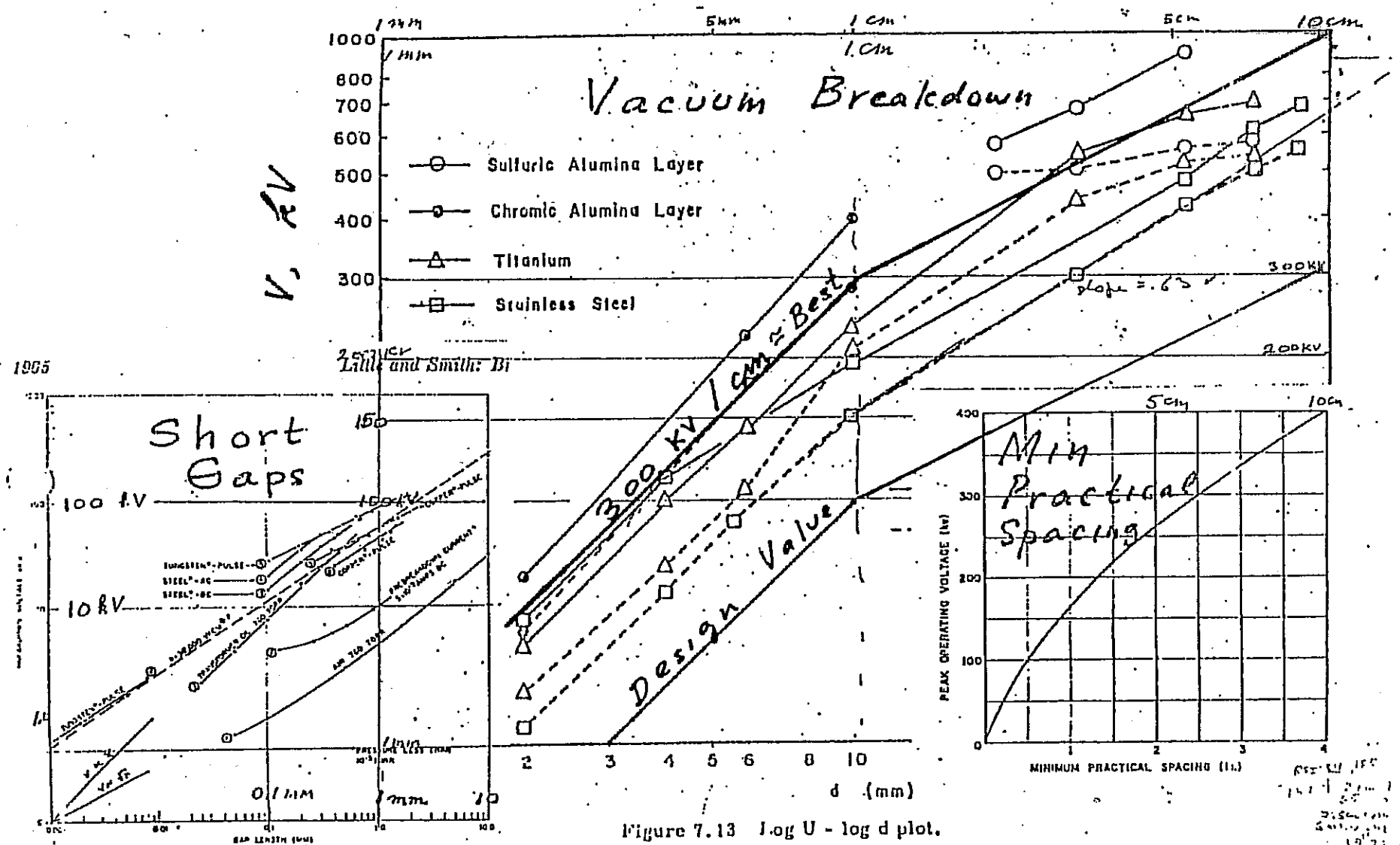
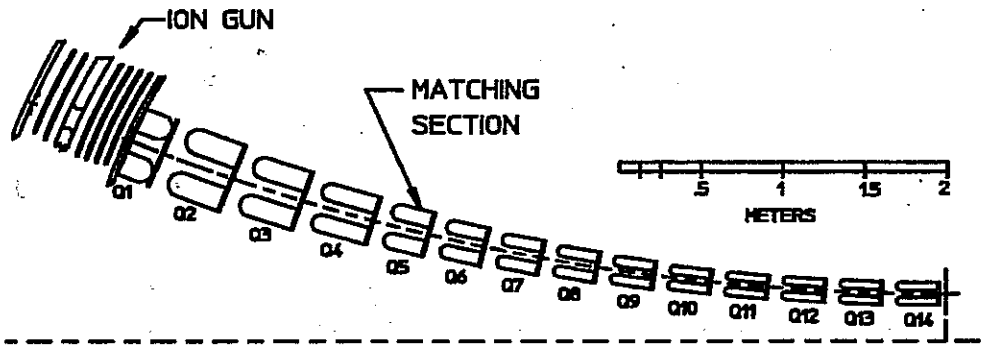


Fig. 1. Breakdown voltage-vs.-gap length for uniform-field and near-uniform-field geometry. Numbers on curves indicate the

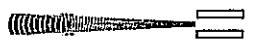
Figure 7.13 Log U - log d plot.

MULTIPLE BEAMLET INJECTORS CAN HAVE HIGHER CURRENT DENSITY
DECREASING SIZE OF INJECTOR

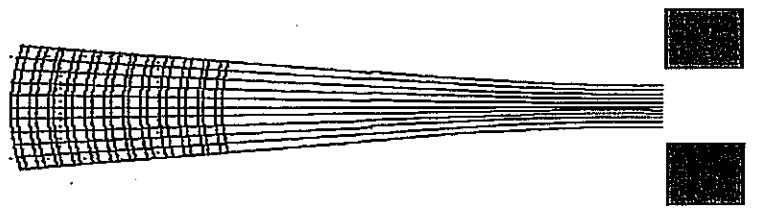
traditional design using single large diameter source →



advanced design using multiple beamlets →



Each beamlet carries higher current density; But merging beamlets increases thermal spread.



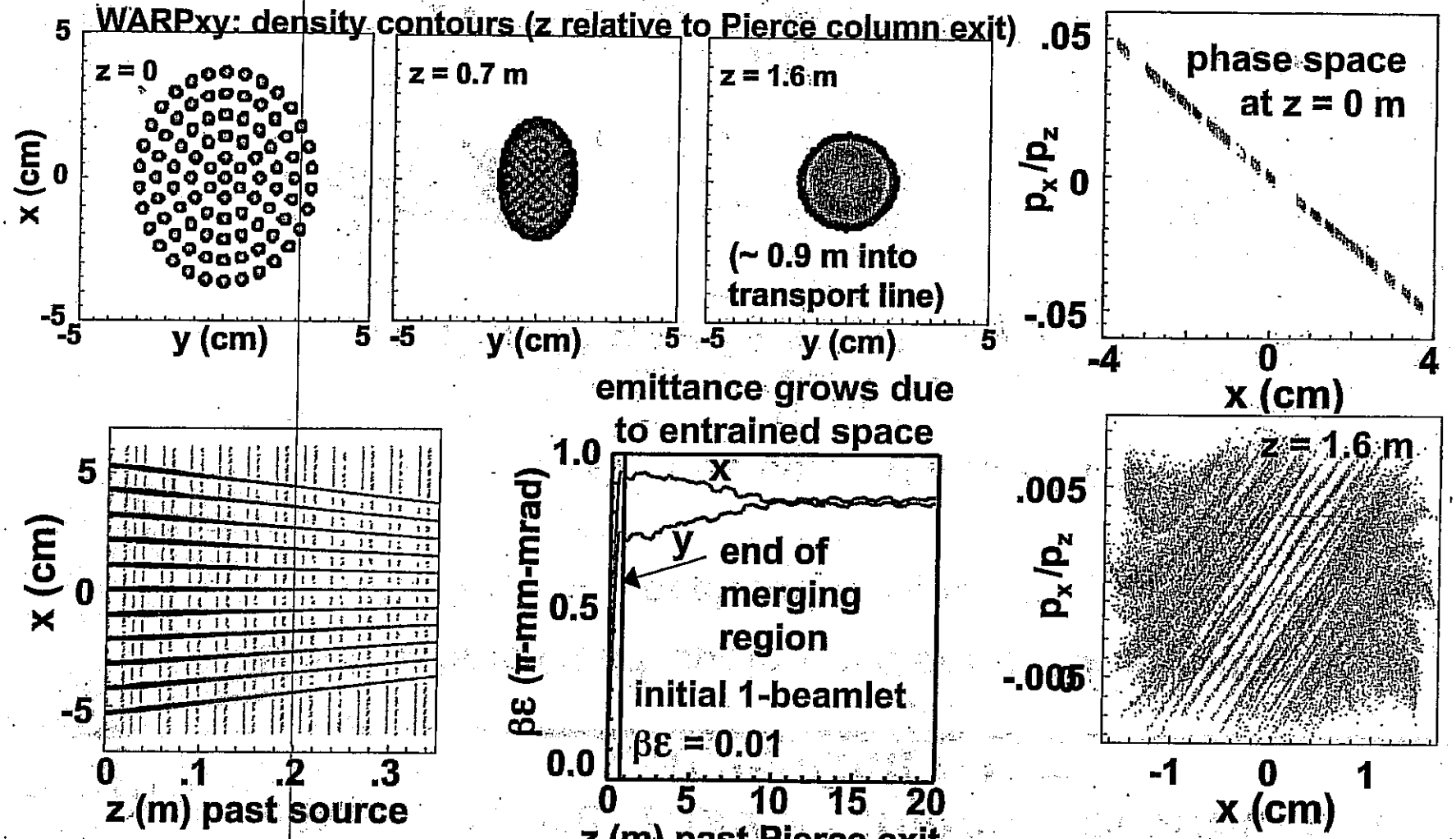
Child-Langmuir $J_{CL} \propto \frac{V^{3/2}}{d^2}$

Breakdown limit $V \propto d^{1.0\ to\ 0.5}$

$$J \propto V^{-1/2\ to\ -5/2} \propto d^{-1/2\ to\ -5/4}$$

Merge and match beamlets into an ESQ channel

Simulations of merging-beamlet injector

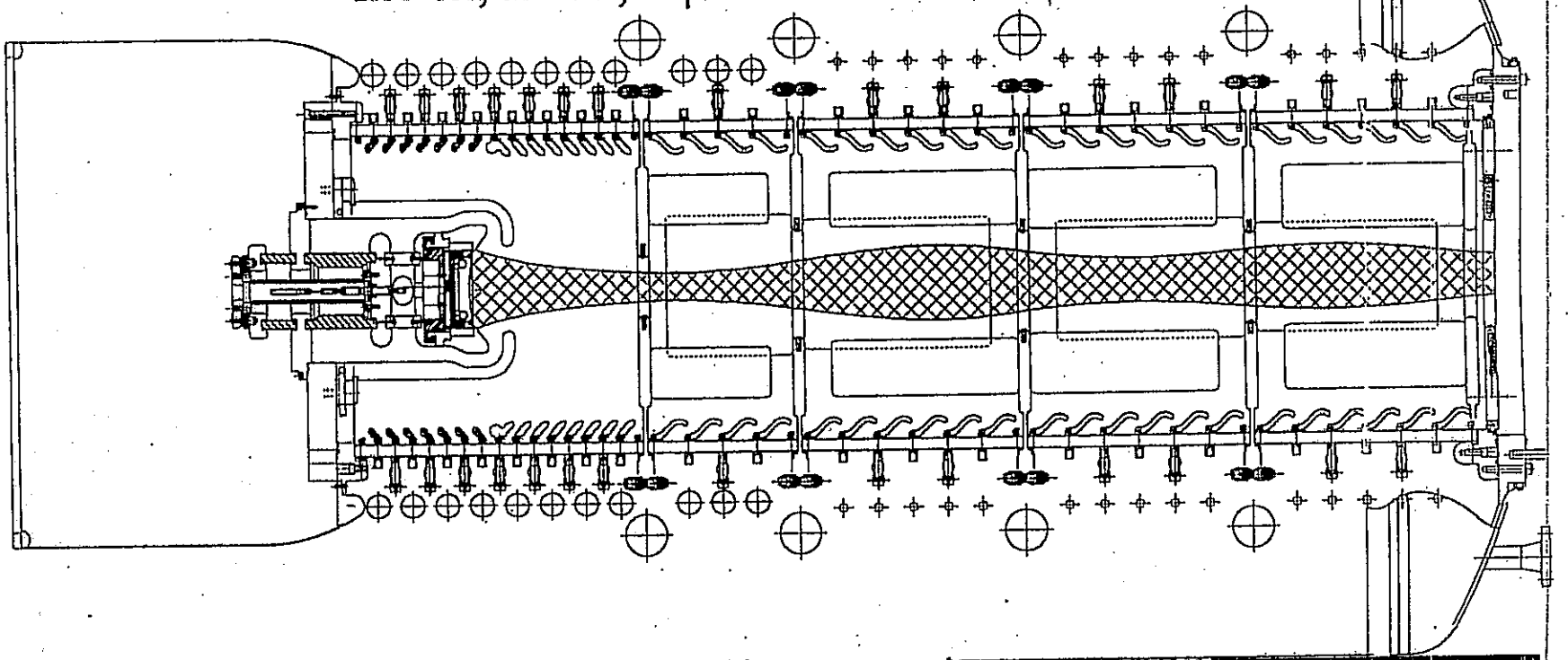


from D.P. GROTE, E. HENESTROZA, J.W. KWAN, "DESIGN & SIMULATION OF THE MULTIBEAMLET INJECTOR FOR A HIGH CURRENT ACCELERATOR" SUBMITTED TO NSTAB (2002)

0.8 Ampere, 2 MV K^+ Injector produced a λ=0.25μC/m beam

Electrostatic Quadrupole Accelerator for simultaneous focusing and acceleration of ion beams to 2 MV.

2.5 m, 2 MV, 2 μs column and ϕ6.7" source



Filmed for ease of reference

SCALING OF BRIGHTNESS IN INJECTORS

$$\epsilon_N = 4\beta\langle x^2\rangle^{1/2}\langle x'^2\rangle^{1/2} = \frac{4}{c}\left(\frac{v_b}{2}\right)\langle v_x^2\rangle^{1/2}$$

$$\frac{1}{2}mv_x^2 = \frac{1}{2}kT$$

$$\epsilon_N = 2 r_b \sqrt{\frac{kT}{mc^2}}$$

$$\Rightarrow \quad B = \frac{I}{\epsilon_N^2} = \frac{\pi J}{4(kT/mc^2)} \sim \frac{J}{T}$$

$\Rightarrow$ FOR HIGH BRIGHTNESS & HIGH CURRENT MAY WISH TO ACCELERATE MANY BEAMLETS AND THEN MERGE TO FORM SINGLE BEAM.

National® Brand 42-182 100 SHEETS Made in U.S.A.

MANY ISSUES NOT DISCUSSED HERE!

- SOURCES
- ELECTRON TRAPPING
- CONVERGING BEAMS
- MATCHING TO AN ESQ (e.g.)
- rf
- ...