Transverse Equilibrium Distribution Functions*

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Transverse Equilibrium Distribution Functions: Outline

Transverse Equilibrium Dist. Functions: Detailed Outline

Detailed Outline - 2

Detailed Outline - 3

S1: Vlasov Model: Transverse Vlasov model for a coasting, single species beam with electrostatic self-fields propagating in an applied focusing lattice:

$\mathbf{x}_\perp, \ \mathbf{x}'_\perp$ transverse particle coordinate, angle

$q, \ m$ charge, mass

$\gamma_b, \ \beta_b$ axial relativistic factors

$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle distribution

$H_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle Hamiltonian

Vlasov Equation (see J.J. Barnard, Introductory Lectures):

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

Particle Equations of Motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Hamiltonian (see S.M. Lund, lectures on *Transverse Particle Equations of Motion*):

$$H_\perp = \frac{1}{2}\mathbf{x}'^{\,2}_\perp + \frac{1}{2}\kappa_x(s)x^2 + \frac{1}{2}\kappa_y(s)y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Poisson Equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\epsilon_0}\int d^2\mathbf{x}'_\perp \ f_\perp$$

+ boundary conditions on ϕ

Hamiltonian expression of the Vlasov equation:

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

$$= \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

Using the equations of motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} = \mathbf{x}'_\perp$$

$$\frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp} = -\left(\kappa_x x\hat{\mathbf{x}} + \kappa_y y\hat{\mathbf{y}} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right)$$

$$\boxed{\frac{\partial f_\perp}{\partial s} + \mathbf{x}'_\perp\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \left(\kappa_x x\hat{\mathbf{x}} + \kappa_y y\hat{\mathbf{y}} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right)\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0}$$

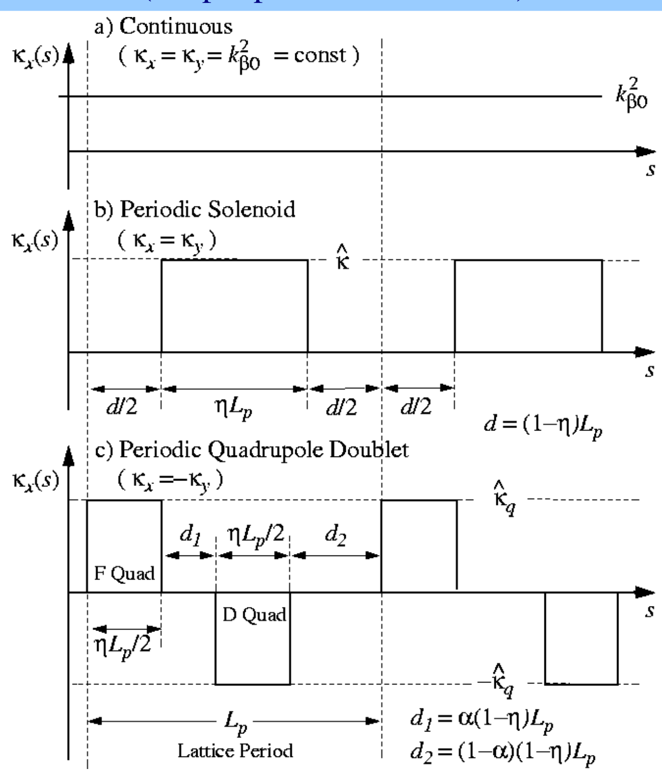
In formal dynamics, a "Poisson Bracket" notation is frequently employed:

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

$$\equiv \frac{\partial f_\perp}{\partial s} + \{H_\perp, f_\perp\} = 0$$

Poisson Bracket

Review: Focusing lattices, continuous and periodic
(simple piecewise constant):



Lattice Period L_p

Occupancy η
$\eta \in [0, 1]$

Solenoid description carried out implicitly in Larmor frame
[see Lund and Bukh, PRST- Accel. and Beams 7, 024801 (2004), Appendix A]

Syncopation Factor α

$$\alpha \in [0, \frac{1}{2}]$$

$$\alpha = \frac{1}{2} \implies FODO$$

Example Hamiltonians:

Continuous focusing $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2 c^2}\phi$$

Solenoidal focusing (in Larmor frame variables) $\kappa_x = \kappa_y = \kappa(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}\kappa \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2 c^2}\phi$$

Quadrupole focusing $\kappa_x = -\kappa_y = \kappa_q(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}\kappa_q x^2 - \frac{1}{2}\kappa_q y^2 + \frac{q}{m\gamma_b^3\beta_b^2 c^2}\phi$$

Review: Undepressed particle phase advance σ_0 is typically employed to characterize the applied focusing strength of periodic lattices:

x-orbit without space-charge satisfies Hill's equation

$$x''(s) + \kappa_x(s)x(s) = 0$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \mathbf{M}_x(s \mid s_i) \cdot \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

$\mathbf{M}_x =$ 2 x 2 Transfer Matrix from $s = s_i$ to s

Undepressed phase advance

$$\boxed{\cos\sigma_{0x} = \frac{1}{2}\text{Tr }\mathbf{M}_x(s_i + L_p|s_i)}$$

Single particle (and centroid) stability requires:

$$\frac{1}{2}\text{Tr }\mathbf{M}_x(s_i + L_p|s_i) < 1 \longrightarrow \boxed{\sigma_{0x} < 180^\circ}$$

[Courant and Snyder, Annals of Phys. **3**, 1 (1958)]

Analogous equations hold in the y-plane

S2: Vlasov Equilibria: Plasma physics-like approach is to resolve the system into an equilibrium + perturbation and analyze stability

Equilibrium constructed from single-particle constants of motion C_i

$$\boxed{f_\perp = f_\perp(\{C_i\}) \quad \text{equilibrium}}$$

$$\frac{d}{ds} f_\perp(\{C_i\}) = \sum_i \frac{\partial f_\perp}{\partial C_i} \cancelto{0}{\frac{dC_i}{ds}} = 0$$

Comments:

- Equilibrium is an exact solution to Vlasov's equation that does not change in 4D phase-space as s advances
  - Projections of the distribution can evolve in s in general cases
- Particle conversation constraints are in the presence of (possibly s-varying) applied and space-charge forces
  - Highly non-trivial!
  - Only one exact solution known for s-varying focusing: the KV distribution to be analyzed shortly in this lecture.

// Example: Continuous focusing $f_\perp = f_\perp(H_\perp)$

$$H_\perp = \frac{1}{2}\mathbf{x}'^{\,2}_\perp + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi \qquad \text{no explicit s dependance}$$

$$\frac{df_\perp}{ds} = \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp}$$

$$= \frac{\partial f_\perp}{\partial H_\perp}\cancelto{0}{\frac{\partial H_\perp}{\partial s}} + \frac{\partial f_\perp}{\partial H_\perp}\left(\frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial H_\perp}{\partial \mathbf{x}_\perp} \cancelto{0}{-} \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\right) = 0$$

Showing that $f_\perp = f_\perp(H_\perp)$ exactly satisfies Vlasov's equation for continuous focusing

//

Typical single particle constants of motion:

Transverse Hamiltonian for continuous focusing:

$$H_\perp = \frac{1}{2}\mathbf{x}'^{\,2}_\perp + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2 c^2}\phi = \text{const}$$

$$k_{\beta 0}^2 = \text{const}$$

Canonical angular momentum for rotationally invariant systems:

$$P_\theta = xy' - yx' = \text{const}$$

(in Larmor frame variables for solenoidal focusing)

Axial kinetic energy for systems with no acceleration:

$$\mathcal{E} = (\gamma_b - 1)mc^2 = \text{const}$$

More on other classes of constraints later ...

Plasma physics approach to beam physics:

Resolve:

$$\boxed{f(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = f_\perp(\{C_i\}) + \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)}$$

equilibrium perturbation $f_\perp \gg |\delta f_\perp|$

and carry out equilibrium + stability analysis

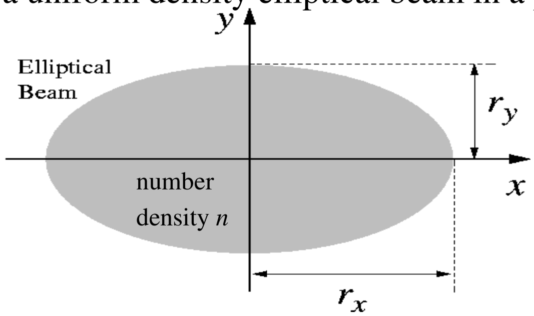
Comments:

- Attraction is to parallel the impressive successes of plasma physics
  - Gain insight into preferred state of nature
- Beams are born off a source and may not be close to an equilibrium condition
  - Appropriate single particle constants of the motion unknown for periodic focusing lattices other than the (unphysical) KV distribution
- Intense beam self-fields and finite radial extent vastly complicate equilibrium description and analysis of perturbations
  - It is not clear if smooth Vlasov equilibria exist in periodic focusing
  - Higher model detail vastly complicates picture!
- If system can be tuned to more closely resemble a relaxed, equilibrium, one might expect less deleterious effects based on plasma physics analogies

S3: The KV Equilibrium Distribution

[Kapchinskij and Vladimirskij, Proc. Int. Conf. On High Energy Accel., 1959]

Assume a uniform density elliptical beam in a periodic focusing lattice



Line-Charge:

$$\lambda = qn(s)\pi r_x(s) r_y(s)$$
$$= \text{const}$$

Perveance:

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2}$$
$$= \text{const}$$

Free-space self field solution within the beam (see Appendix A)

$$\phi = -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{x^2}{(r_x+r_y)r_x} + \frac{y^2}{(r_x+r_y)r_y}\right] + \text{const}$$

Particle equations of motion within the beam (Hill's equation if edge radii given):

$$\boxed{\begin{aligned} x''(s) + \left\{\kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)}\right\} x(s) = 0 \\ y''(s) + \left\{\kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)}\right\} y(s) = 0 \end{aligned}}$$

If we regard the envelope radii as specified functions of s, then these equations of motion are Hill's equations familiar from elementary accelerator physics:

$$\boxed{\begin{aligned} x''(s) + \kappa_x^{\text{eff}}(s)x(s) = 0 \\ y''(s) + \kappa_y^{\text{eff}}(s)y(s) = 0 \\ \kappa_x^{\text{eff}}(s) = \kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)} \\ \kappa_y^{\text{eff}}(s) = \kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)} \end{aligned}}$$

Suggests Procedure:

- Calculate Courant-Snyder invariants under assumptions made
- Construct a distribution function of Courant-Snyder invariants that generates the uniform density elliptical beam projection assumed
  - Nontrivial step: guess and show that it works
- Resulting distribution will be an equilibrium that does not change 4D form as a function of s

Review (1): The Courant-Snyder invariant of Hill's equation

[Courant and Snyder, Annl. Phys. **3**, 1 (1958)]

Hill's equation describes a zero space-charge particle orbit in linear applied focusing fields:

$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

As a consequence of Floquet's theorem, the solution can be cast in phase-amplitude form:

$$x(s) = A_i w(s) \cos\psi(s)$$

where $w(s)$ is the periodic solution to

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s) \qquad w(s) > 0$$

$\psi(s)$ is a phase function given by

$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

A_i and ψ_i are constants set by initial conditions at $s = s_i$

Review (2): The Courant-Snyder invariant of Hill's equation

From this formulation it follows immediately that

$$x(s) = A_i w(s) \cos\psi(s)$$

$$x'(s) = A_i w'(s) \cos\psi(s) + \frac{A_i}{w(s)} \sin\psi(s)$$

or

$$\frac{x}{w} = A_i \cos\psi$$

$$wx' - w'x = A_i \sin\psi$$

square and add equations to obtain the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}}$$

- Simplifies interpretation of dynamics
- Extensively used in accelerator physics

Phase-amplitude description of particles evolving within a uniform density beam:

Phase-amplitude form of x-orbit equations:

$$x(s) = A_{xi} w_x(s) \cos\psi_x(s)$$

$$x'(s) = A_{xi} w_x'(s) \cos\psi_x(s) - \frac{A_{xi}}{w_x(s)} \sin\psi_x(s)$$

initial conditions yield: $(s = s_i)$

$$A_{xi} = \text{const}$$

$$\psi_{xi} = \psi_x(s = s_i) = \text{const}$$

where

$$\boxed{\begin{gathered} w_x''(s) + \kappa_x(s) w_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)} w_x(s) - \frac{1}{w_x^3(s)} = 0 \\ w_x(s+L_p) = w_x(s) \qquad w_x(s) > 0 \\ \psi_x(s) = \psi_{xi} + \int_{s_i}^{s} \frac{d\tilde{s}}{w_x^2(\tilde{s})} \end{gathered}}$$

identifies the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_{xi}^2 = \text{const}}$$

Analogous equations hold for the *y*-plane

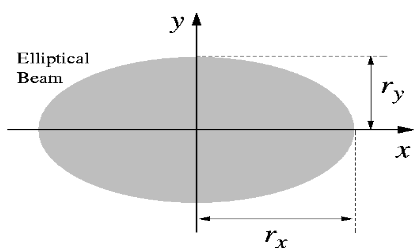
The KV envelope equations:

Define *maximum* Courant-Snyder invariants:

$$\varepsilon_x \equiv \text{Max}(A_{xi}^2)$$

$$\varepsilon_y \equiv \text{Max}(A_{yi}^2)$$

These values must correspond to the beam-edge:



$$r_x(s) = \sqrt{\varepsilon_x}\, w_x(s)$$

$$r_y(s) = \sqrt{\varepsilon_y}\, w_y(s)$$

The equations for w_x and w_y can then be rescaled to obtain the familiar KV envelope equations for the matched beam envelope

$$\boxed{\begin{gathered} r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0 \\ r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0 \\ r_x(s+L_p) = r_x(s) \qquad r_x(s) > 0 \\ r_y(s+L_p) = r_y(s) \qquad r_y(s) > 0 \end{gathered}}$$

Use variable rescalings to denote x- and y-plane Courant-Snyder invariants as:

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w'_x x)^2 = A_{xi}^2 = \text{const}$$

$$\boxed{\begin{aligned} \left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r'_x x}{\varepsilon_x}\right)^2 = C_x = \text{const} \\ \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r'_y y}{\varepsilon_y}\right)^2 = C_y = \text{const} \end{aligned}}$$

Kapchinskij and Vladimirskij constructed a delta-function distribution of a linear combination of these Courant-Snyder invariants that generates the correct uniform density elliptical beam needed for consistency with the assumptions:

$$f_\perp = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[C_x + C_y - 1\right]$$

- Delta function means the sum of the x- and y-invariants is a constant
- Other forms would not generate the needed uniform density elliptical beam projection

The KV equilibrium is constructed from the Courant-Snyder invariants:

KV equilibrium distribution:

$$\boxed{\begin{aligned} f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta \Bigg[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r'_x x}{\varepsilon_x}\right)^2 + \\ \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r'_y y}{\varepsilon_y}\right)^2 - 1 \Bigg] = \text{const} \\ \delta(x) = \text{ Dirac delta function} \end{aligned}}$$

This distribution generates (see proof in Appendix B) the correct uniform density elliptical beam:

$$\boxed{n = \int d^2x'_\perp \; f_\perp \; = \begin{cases} \frac{\lambda}{q\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}}$$

Obtaining this form consistent with the assumptions demonstrates full self-consistency of the KV equilibrium distribution.

- Full 4-D form of the distribution does not evolve in s
- Projections of the distribution can (and generally do!) evolve in s

Comment on notation of integrals:

- 2^{nd} forms useful for systems with azimuthal spatial or annular symmetry

Spatial

$$\int d^2x_\perp \cdots \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \cdots$$

$$= \int_0^\infty dr\, r \int_{-\pi}^{\pi} d\theta \cdots$$

Cylindrical Coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Angular

$$\int d^2x'_\perp \cdots \equiv \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \cdots$$

$$= \int_0^\infty dr'\, r' \int_{-\pi}^{\pi} d\theta' \cdots$$

Angular Cylindrical Coordinates:

$$x' = r'\cos\theta'$$
$$y' = r'\sin\theta'$$

Comment on notation of integrals (continued):

Axisymmetry simplifications

Spatial: for some function $f(\mathbf{x}_\perp^2) = f(r^2)$

$$\int d^2x_\perp\, f(\mathbf{x}_\perp^2) = 2\pi \int_0^\infty dr\, r f(r^2)$$
$$= \pi \int_0^\infty dr^2\, f(r^2)$$
$$= \pi \int_0^\infty dw\, f(w)$$

Cylindrical Coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$w = r^2$$

Angular: for some function $g(\mathbf{x}_\perp'^2) = g(r'^2)$

$$\int d^2x'_\perp\, g(\mathbf{x}_\perp'^2) = 2\pi \int_0^\infty dr'\, r' g(r'^2)$$
$$= \pi \int_0^\infty dr'^2\, g(r'^2)$$
$$= \pi \int_0^\infty du\, g(u)$$

Angular Cylindrical Coordinates:

$$x' = r'\cos\theta'$$
$$y' = r'\sin\theta'$$

$$u = r'^2$$

Moments of the KV distribution can be calculated directly from the distribution to further aid interpretation:

Full 4D average:
$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \ \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \ f_\perp}$$

Restricted angle average:
$$\langle \cdots \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \ \cdots f_\perp}{\int d^2x'_\perp \ f_\perp}$$

Envelope edge radius:
$$r_x = 2\langle x^2 \rangle_\perp^{1/2}$$

rms edge emittance (maximum Courant-Snyder invariant):
$$\varepsilon_x = 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp^2]^{1/2}$$

Coherent flows (within the beam, zero otherwise):
$$\langle x' \rangle_{\mathbf{x}'_\perp} = r'_x \frac{x}{r_x}$$

Angular spread (x-temperature, within the beam, zero otherwise):
$$T_x \equiv \langle (x' - \langle x' \rangle_{\mathbf{x}'_\perp})^2 \rangle_{\mathbf{x}'_\perp} = \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)$$

Summary of 1st and 2nd order moments of the KV distribution:

Moment	Value
$\int d^2x'_\perp \ x' f_\perp$	$r'_x \frac{x}{r_x} n$
$\int d^2x'_\perp \ y' f_\perp$	$r'_y \frac{y}{r_y} n$
$\int d^2x'_\perp \ x'^2 f_\perp$	$\left[r'^2_x \frac{x^2}{r_x^2} + \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] n$
$\int d^2x'_\perp \ y'^2 f_\perp$	$\left[r'^2_y \frac{y^2}{r_y^2} + \frac{\varepsilon_y^2}{2r_y^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] n$
$\int d^2x'_\perp \ xx' f_\perp$	$\frac{r'_x}{r_x} x^2 n$
$\int d^2x'_\perp \ yy' f_\perp$	$\frac{r'_y}{r_y} y^2 n$
$\int d^2x'_\perp \ (xy' - yx') f_\perp$	0
$\langle x^2 \rangle_\perp$	$\frac{r_x^2}{4}$
$\langle y^2 \rangle_\perp$	$\frac{r_y^2}{4}$
$\langle x'^2 \rangle_\perp$	$\frac{r'^2_x}{4} + \frac{\varepsilon_x^2}{4r_x^2}$
$\langle y'^2 \rangle_\perp$	$\frac{r'^2_y}{4} + \frac{\varepsilon_y^2}{4r_y^2}$
$\langle xx' \rangle_\perp$	$\frac{r_x r'_x}{4}$
$\langle yy' \rangle_\perp$	$\frac{r_y r'_y}{4}$
$\langle xy' - yx' \rangle_\perp$	0
$16[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp^2]$	ε_x^2
$16[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle yy' \rangle_\perp^2]$	ε_y^2

All 1st and 2nd order moments not listed vanish, i.e.,

$$\int d^2x'_\perp \ xy f_\perp = 0$$

$$\langle xy \rangle_\perp = 0$$

Canonical transformation illustrates KV distribution structure:
[Davidson, Physics of Nonneutral Plasmas, Addison-Wesley (1990), and Appendix B]

Phase-space transformation:

$$\boxed{\begin{aligned} X &= \frac{\sqrt{\varepsilon_x}}{r_x} x \\ X' &= \frac{r_x x' - r_x' x}{\sqrt{\varepsilon_x}} \end{aligned}}$$

$$dx\,dy = \frac{r_x r_y}{\sqrt{\varepsilon_x \varepsilon_y}} dX\,dY$$

$$dx'\,dy' = \frac{\sqrt{\varepsilon_x \varepsilon_y}}{r_x r_y} dX'\,dY'$$

$$dx\,dy\,dx'\,dy' = dX\,dY\,dX'\,dY'$$

Courant-Snyder invariants in the presence of beam space-charge are then simply:

$$\boxed{X^2 + X'^2 = \text{const}}$$

and the KV distribution takes the simple, symmetrical form:

$$\boxed{f_\perp(x, y, x', y', s) = f_\perp(X, Y, X', Y') = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[\frac{X^2 + X'^2}{\varepsilon_x} + \frac{Y^2 + Y'^2}{\varepsilon_y} - 1\right]}$$

from which the density and other projections can be more easily (see Appendix B) calculated:

$$n = \int d^2 x'_\perp\; f_\perp \;= \frac{\lambda}{q\pi r_x r_y} \int_0^\infty dU^2\;\delta\left[U^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right]$$

$$= \begin{cases} \frac{\lambda}{q\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

KV Envelope equation

The envelope equation reflects low-order force balances

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

	Applied Focusing	Space-Charge Defocusing	Thermal Defocusing
Terms:	Lattice	Perveance	Emittance

Matched Solution

$$r_x(s + L_p) = r_x(s)$$

$$r_y(s + L_p) = r_y(s)$$

- Envelope equation is a projection of the 4D invariant distribution
- Most important basic design equation for transport lattices with high space-charge intensity
  - Simplest consistent design equations incorporating applied focusing, space-charge defocusing, and thermal defocusing forces
  - Starting point of almost all practical machine design!
- Instabilities of envelope equations are well understood and real (to be covered: lectures on *Centroid and Envelope Description of Beams*)
  - Must be avoided for reliable machine operation

The matched solution to the KV envelope equations reflects the symmetry of the focusing lattice and must in general be calculated numerically

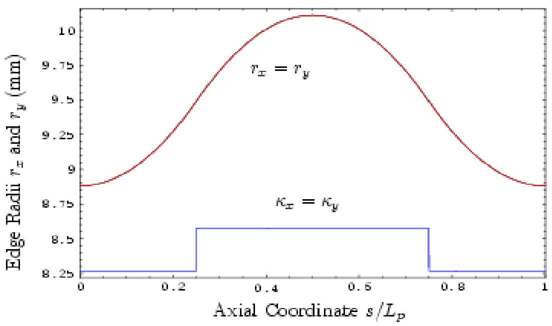
$$\boxed{\begin{aligned} r_x(s+L_p) &= r_x(s) \\ r_y(s+L_p) &= r_y(s) \\ \varepsilon_x &= \varepsilon_y \end{aligned}}$$

Parameters

$$L_p = 0.5\ \text{m}, \quad \sigma_0 = 80^\circ, \quad \eta = 0.5$$
$$\varepsilon_x = 50\ \text{mm-mrad}$$
$$\sigma/\sigma_0 = 0.2$$

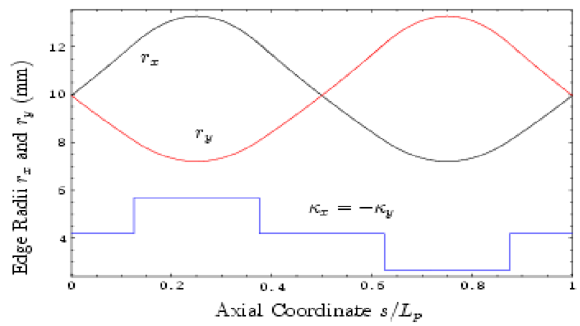
Solenoidal Focusing

$(Q = 6.6986 \times 10^{-4})$



FODO Quadrupole Focusing

$(Q = 6.5614 \times 10^{-4})$



The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

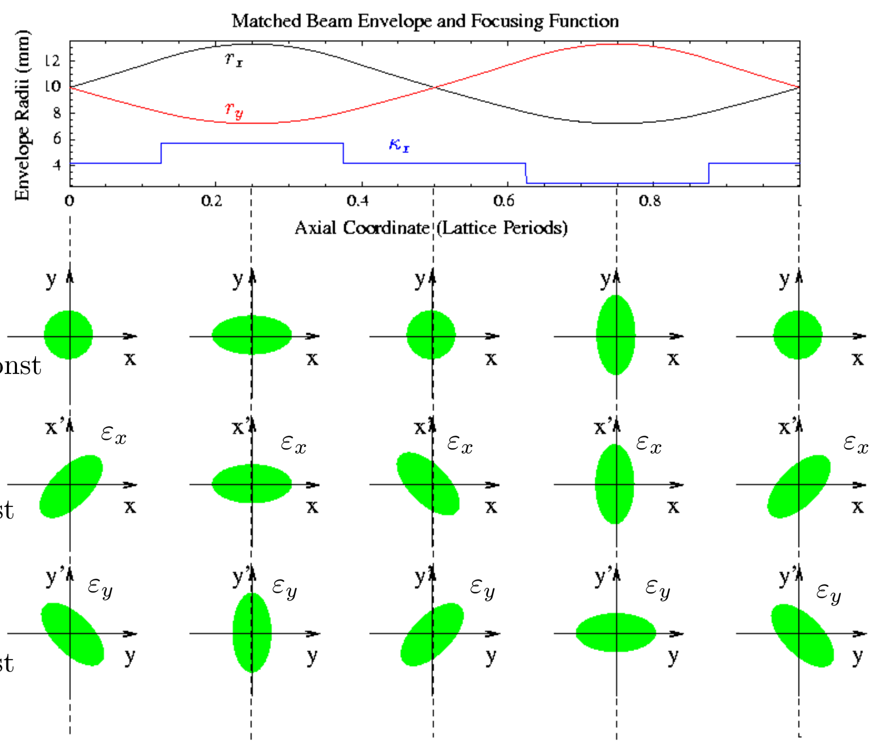
Beam symmetries of a matched KV equilibrium beam in a periodic FODO transport lattice

Projection

x-y
area: $\pi r_x r_y \neq$ const

x-x'
area: $\pi\varepsilon_x =$ const
(CS Invariant)

y-y'
area: $\pi\varepsilon_y =$ const
(CS Invariant)



KV model shows that particle orbits in the presence of space-charge can be strongly modified – space charge slows the orbit response:

Matched envelope:

$$r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0$$
$$r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0$$
$$r_x(s + L_p) = r_x(s) \qquad r_x(s) > 0$$
$$r_y(s + L_p) = r_y(s) \qquad r_y(s) > 0$$

Equation of motion for x-plane "depressed" orbit in the presence of space-charge:

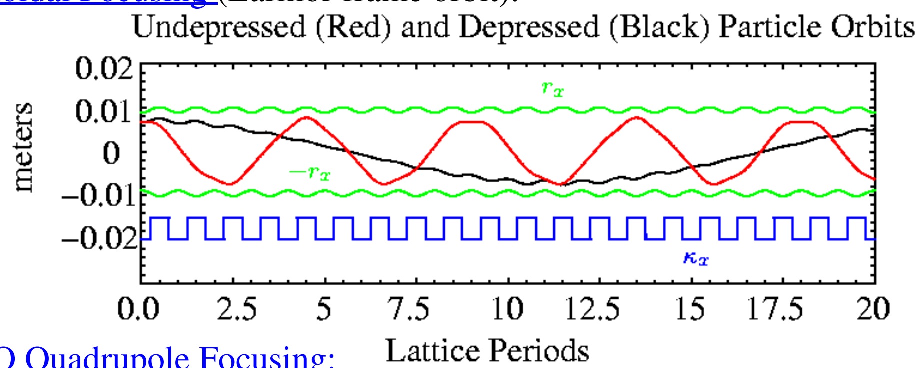
$$x''(s) + \kappa_x(s) x(s) - \frac{2Q}{[r_x(s) + r_y(s)] r_x(s)} x(s) = 0$$

All particles have the *same value* of depressed phase advance:

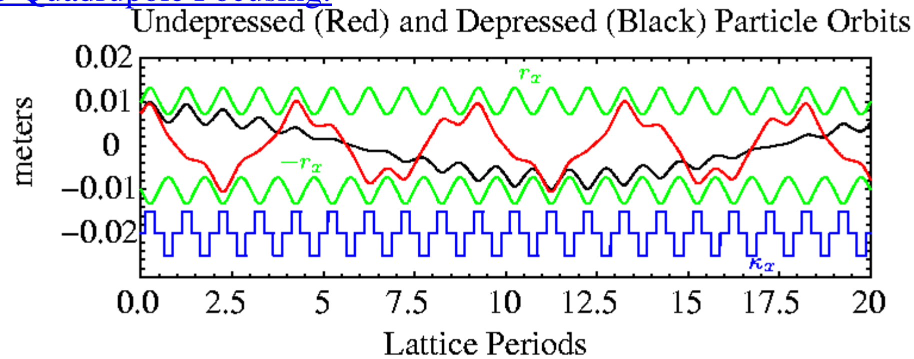
$$\sigma_x \equiv \psi_x(s_i + L_p) - \psi_x(s_i) = \varepsilon_x \int_{s_i}^{s_i + L_p} \frac{ds}{r_x^2(s)}$$

Depressed particle x-plane orbits within a matched KV beam in a periodic FODO quadrupole channel for the matched beams previously shown

Solenoidal Focusing (Larmor frame orbit):



FODO Quadrupole Focusing:



Depressed particle phase advance provides a convenient measure of space-charge strength

For simplicity take (plane symmetry in average focusing and emittance)

$$\sigma_{0x} = \sigma_{0y} \equiv \sigma_0 \qquad \varepsilon_x = \varepsilon_y \equiv \varepsilon$$

Depressed phase advance within a matched beam

$$\boxed{\sigma = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)} = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_y^2(s)}}$$

$$\lim_{Q\to 0} \sigma = \sigma_0$$

Normalized space charge strength

$$\boxed{0 \leq \sigma/\sigma_0 \leq 1}$$

$\sigma/\sigma_0 \to 0$ Cold Beam (space-charge dominated) $\varepsilon \to 0$

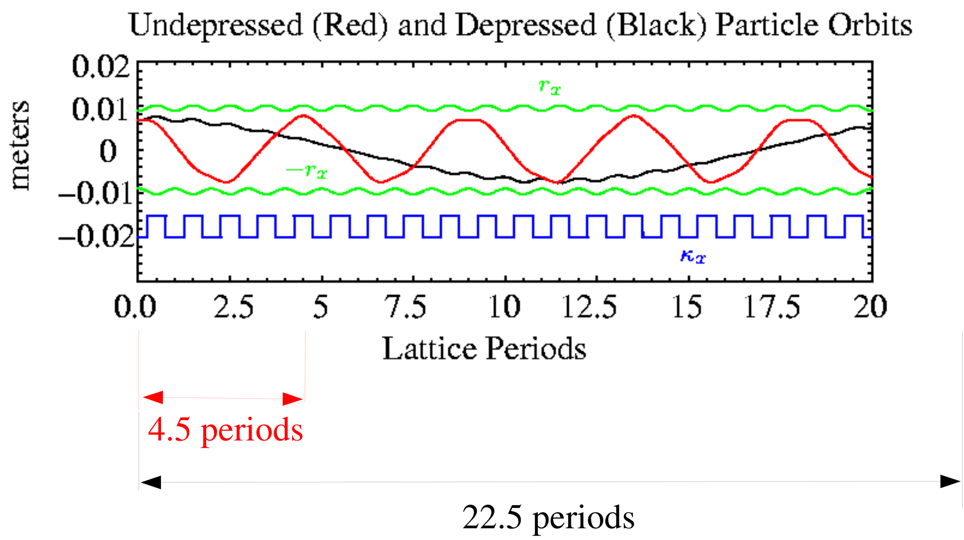
$\sigma/\sigma_0 \to 1$ Warm Beam (kinetic dominated) $Q \to 0$

For example matched envelope presented earlier:

			repeat periods
Undepressed phase advance:	$\sigma_0 = 80^\circ$		4.5
Depressed phase advance:	$\sigma = 16^\circ$	$\to \sigma/\sigma_0 = 0.2$	22.5

Periods for 360 degree phase advance

Solenoidal Focusing (Larmor frame orbit):



The rms equivalent beam model helps interpret general beam evolution in terms of an "equivalent" local KV distribution

For the same focusing lattice, replace any beam charge $\rho(x,y)$ density by a uniform density KV beam in each axial slice (s) using averages calculated from the actual "real" beam distribution with:

$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp} \qquad f_\perp = \text{ real distribution}$$

rms equivalent beam:

Quantity	KV Equiv.	Calculated from Distribution
Perveance	Q	$= q^2 \int d^2x_\perp \int d^2x'_\perp \; f_\perp \; / [2\pi\epsilon_0 \gamma_b^3 \beta_b^2 c^2]$
x-edge radius	r_x	$= 2\langle x^2 \rangle_\perp^{1/2}$
y-edge radius	r_y	$= 2\langle y^2 \rangle_\perp^{1/2}$
x-emittance	ε_x	$= 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp]^{1/2}$
y-emittance	ε_y	$= 4[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle yy' \rangle_\perp]^{1/2}$

Comments on rms equivalent beam concept:

- The emittances will generally evolve in s
  - Means that the equivalency must be recalculated in every slice as the emittances evolve
  - For reasons to be analyzed later (lectures on Kinetic Stability of Beams), this evolution is often small
- Concept is highly useful
  - KV equilibrium properties well understood and are approximately correct to model lowest order "real" beam properties

Sacherer expanded the concept of rms equivalency by showing that the equivalency works exactly for beams with elliptic symmetry space-charge [Sacherer, IEEE Trans. Nucl. Sci. 18, 1101 (1971), J.J. Barnard, Intro. Lectures]

For any beam with elliptic symmetry charge density in each transverse slice:

$$\boxed{\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)}$$

Based on:

$$\langle x \frac{\partial \phi}{\partial x} \rangle_\perp = -\frac{\lambda}{4\pi\epsilon_0} \frac{r_x}{r_x + r_y}$$

see J.J. Barnard intro. lectures

the KV envelope equations

$$\boxed{\begin{aligned} r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2(s)}{r_x^3(s)} &= 0 \\ r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2(s)}{r_y^3(s)} &= 0 \end{aligned}}$$

remain valid when (averages taken with the full distribution):

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const} \qquad \lambda = q\int d^2x_\perp\, \rho \;=\; \text{const}$$

$$r_x = 2\langle x^2 \rangle_\perp^{1/2} \qquad \varepsilon_x = 4[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2}$$

$$r_y = 2\langle y^2 \rangle_\perp^{1/2} \qquad \varepsilon_y = 4[\langle y^2\rangle_\perp \langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2]^{1/2}$$

The emittances must, in general, evolve in *s* under this model

(see SM Lund lectures on *Transverse Kinetic Stability*)

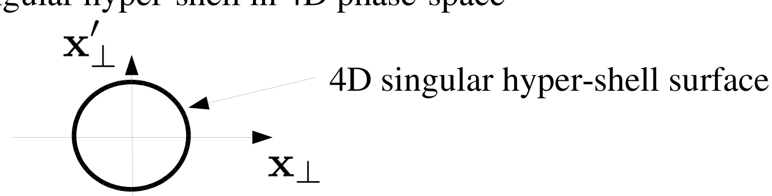
Further comments on the KV equilibrium: Distribution Structure

Equilibrium distribution:

$$\boxed{f_\perp \sim \delta[\text{Courant-Snyder invariants}]}$$

Forms a highly singular hyper-shell in 4D phase-space

Schematic:

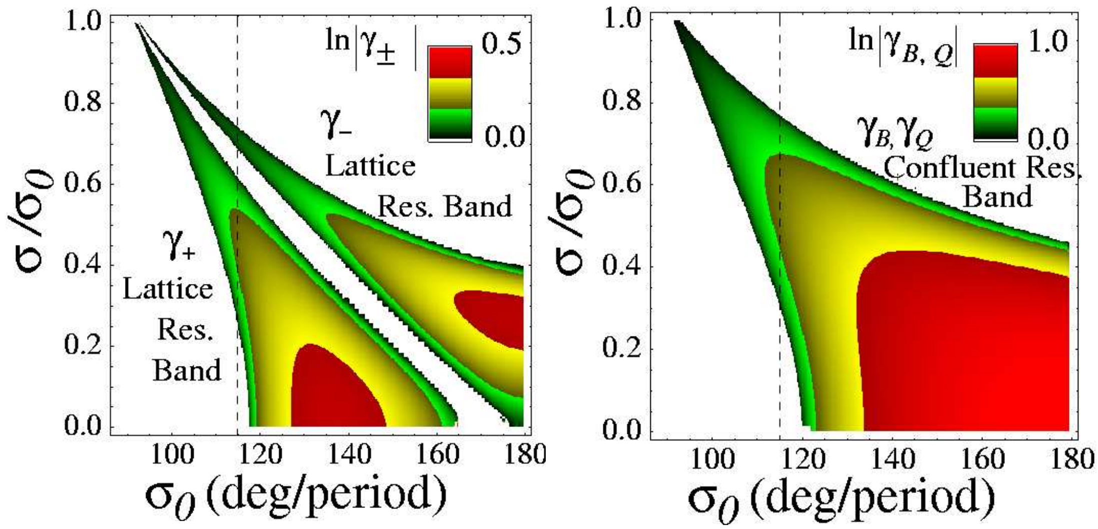


- Singular distribution has large "Free-Energy" to drive many instabilities
  - Low order envelope modes are physical and highly important (see lectures on *Centroid and Envelope Descriptions of Beams*)
- Perturbative analysis shows strong collective instabilities
  - Hofmann, Laslett, Smith, and Haber, Part. Accel. **13**, 145 (1983)
  - Higher order instabilities (collective modes) have unphysical aspects due to (delta-function) structure of distribution and must be applied with care (see lectures on *Kinetic Stability of Beams*)
  - Instabilities can cause problems if the KV distribution is employed as an initial beam state in self-consistent simulations

Preview: lecture on *Centroid and Envelope Descriptions of Beams*
Instability bands of the KV envelope equation are well understood in periodic focusing channels and must be avoided in machine operation

Envelope Mode Instability Growth Rates

Solenoid (η = 0.25) Quadrupole FODO (η = 0.70)

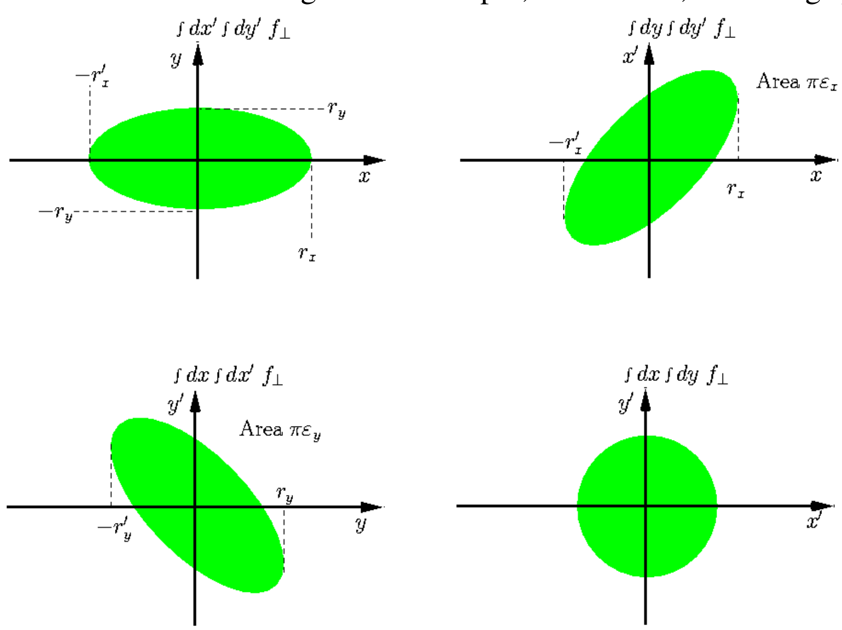


[S.M. Lund and B. Bukh, PRSTAB 024801 (2004)]

Further comments on the KV equilibrium: 2D Projections

All 2D projections of the KV distribution are uniformly filled ellipses

- Not very different from what is often observed in experimental measurements and self-consistent simulations of stable beams with strong space-charge
- Falloff of distribution at "edges" can be rapid, but smooth, for strong space-charge

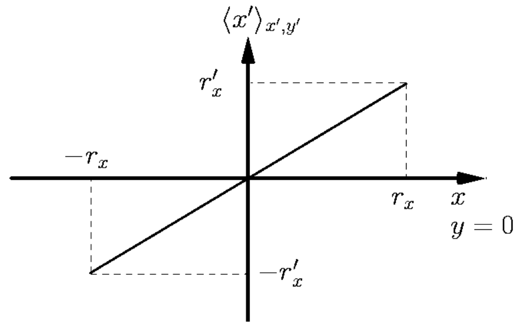


Further comments on the KV equilibrium:
Angular Spreads: Coherent and Incoherent

Angular spreads within the beam:

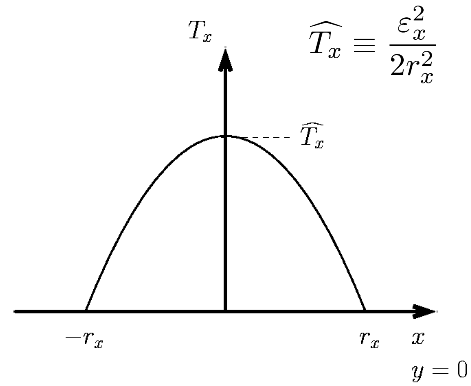
Coherent (flow):

$$\langle x' \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \; x'_\perp \; f_\perp}{\int d^2x'_\perp \; f_\perp} = r'_x \frac{x}{r_x}$$



Incoherent (temperature):

$$\langle (x' - r'_x x/r_x)^2 \rangle_{\mathbf{x}'_\perp} = \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)$$



- Coherent flow required for periodic focusing to conserve charge
- Temperature must be zero at the beam edge since the distribution edge is sharp
- Parabolic temperature profile is consistent with linear grad P pressure forces in a fluid model interpretation of the (kinetic) KV distribution

Further comments on the KV equilibrium:

The KV distribution is the *only* known exact equilibrium solution for linear periodic focusing channels that is valid for finite space-charge:

- Low order properties of the distribution are physically appealing
- Illustrates relevant Courant-Snyder invariants in simple form
  - Later arguments demonstrate that these invariants should be a reasonable approximation for beams with strong space charge

Strong Vlasov instabilities associated with the KV model render the distribution inappropriate for use in high levels of detail:

- Instabilities are not all physical and render interpretation of results difficult
  - Difficult to separate physical from nonphysical effects in simulations

Possible Research Problem (unsolved in 40+ years!):

Can a valid Vlasov equilibrium be constructed for a smooth, nonuniform density distribution in a linear, periodic focusing channel?

- Not clear what invariants can be used or if any can exist
  - Nonexistence proof would also be significant
- Lack of a smooth equilibrium would not imply that real machines cannot work!

Because of a lack of theory for a smooth, self-consistent distribution that would be more physically appealing than the KV distribution we will examine smooth distributions in the idealized continuous focusing limit (after an analysis of the continuous limit of the KV theory):

- Allows more classic "plasma physics" like analysis
- Illuminates physics of intense space charge
- Lack of continuous focusing in the laboratory will prevent over generalization of results obtained

S4: Continuous Focusing limit of the KV Equilibrium Distribution

Continuous focusing, symmetric beam

$$\boxed{\begin{aligned} &\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const} \\ &\varepsilon_x = \varepsilon_y \equiv \varepsilon \\ &r_x = r_y \equiv r_b \end{aligned}}$$

Undepressed betatron wavenumber

envelope equation reduces to

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0 \longrightarrow \boxed{r_b = \left(\frac{Q + \sqrt{4k_{\beta 0}^2 \varepsilon^2 + Q^2}}{2k_{\beta 0}^2} \right)^{1/2} = \text{const}}$$

Particle orbit in the beam:

$$\mathbf{x}_\perp'' + k_\beta^2 \mathbf{x}_\perp = 0 \qquad k_\beta = \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} = \text{const}$$

Depressed betatron wavenumber

$$\longrightarrow \boxed{\mathbf{x}_\perp(s) = \mathbf{x}_{\perp i} \cos[k_\beta(s - s_i)] + \frac{\mathbf{x}_{\perp i}'}{k_\beta} \sin[k_\beta(s - s_i)]}$$

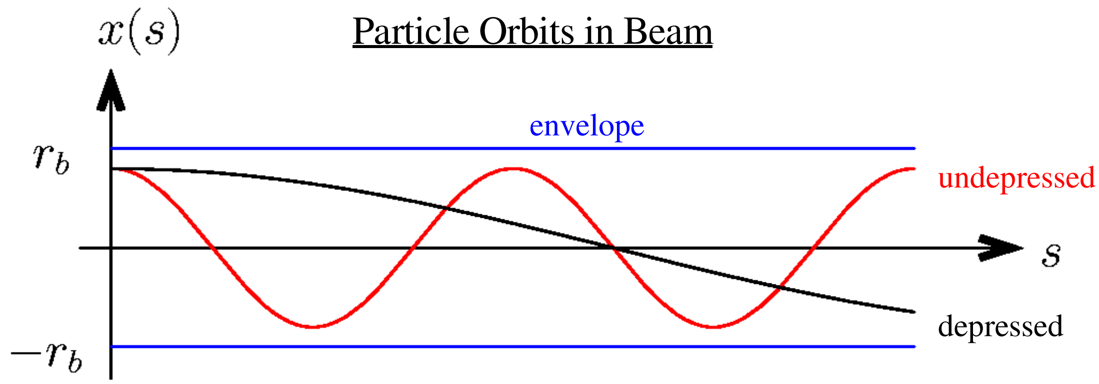
Space-charge tune depression (rate of phase advance same everywhere, L_p arb.)

$$\frac{k_\beta}{k_{\beta 0}} = \frac{\sigma}{\sigma_0} = \left(1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right)^{1/2}$$

$$\underset{\varepsilon \to 0}{0} \leq \frac{\sigma}{\sigma_0} \leq \underset{Q \to 0}{1}$$

Continuous Focusing KV Equilibrium – Undepressed and depressed particle orbits

$$k_\beta = \frac{\sigma}{\sigma_0} k_{\beta 0} \qquad \frac{\sigma}{\sigma_0} = 0.2$$



Much simpler in details than the periodic focusing case, but qualitatively similar in that space-charge "depresses" the rate of particle phase advance

Continuous Focusing KV Beam – Equilibrium Distribution Form

Using

$$\lambda = q\pi \hat{n} r_b^2 \qquad \hat{n} = \text{const} \quad \text{density within the beam}$$

for the beam line charge and

$$\delta(\text{const} \cdot x) = \frac{\delta(x)}{\text{const}}$$

the full elliptic beam KV distribution can be expressed as

$$\boxed{\begin{aligned} f_\perp &= \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right] \\ &= \frac{\hat{n}}{2\pi} \delta(H_\perp - H_{\perp b}) \end{aligned}}$$

where

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}_\perp^2 \qquad \text{-- Hamiltonian}$$

$$= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3 \beta_b^2 c^2}$$

$$H_{\perp b} \equiv \frac{\varepsilon^2}{2r_b^2} = \text{const} \qquad \text{-- Hamiltonian at beam edge}$$

Equilibrium distribution

$$\boxed{f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})}$$

$$H_{\perp b} = \frac{\varepsilon^2}{2r_b^2} = \text{const}$$

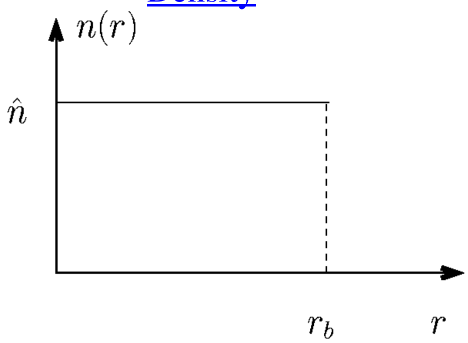
$$\hat{n} = \text{const}$$

then it is straightforward to explicitly calculate (see homework problems)

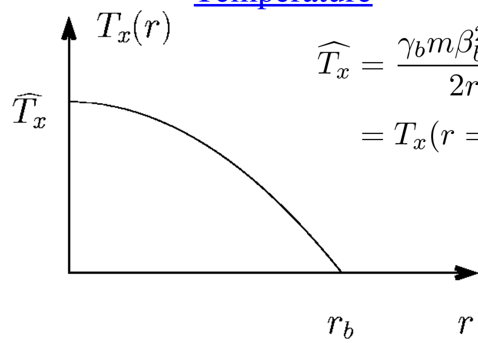
Density: $$n = \int d^2x'_\perp \, f_\perp = \begin{cases} \hat{n}, & 0 \leq r < r_b \\ 0, & r_b < r \end{cases}$$

Temperature: $$T_x = \gamma_b m \beta_b^2 c^2 \frac{\int d^2x'_\perp \, x'^2 f_\perp}{\int d^2x'_\perp \, f_\perp} = \begin{cases} \widehat{T_x}(1 - r^2/r_b^2), & 0 \leq r < r_b \\ 0, & r_b < r \end{cases}$$

Density



Temperature



$$\widehat{T_x} = \frac{\gamma_b m \beta_b^2 c^2 \varepsilon^2}{2r_b^2} = T_x(r = 0)$$

Continuous Focusing KV Beam – Comments

For continuous focusing, $H_\perp$ is a single particle constant of the motion (see problem sets), so it is not surprising that the KV equilibrium form reduces to a delta function form of $f_\perp(H_\perp)$

- Because of the delta-function distribution form, all particles in the continuous focusing KV beam have the same transverse energy with $H_\perp = H_{\perp b} = \text{const}$

Several textbook treatments of the KV distribution derive continuous focusing versions and then just write down (if at all) the periodic focusing version based on Courant-Snyder invariants. This can create a false impression that the KV distribution is a Hamiltonian-type invariant in the general form.

- For non-continuous focusing channels there is no simple relation between Courant-Snyder type invariants and $H_\perp$

S5: Equilibrium Distributions in Continuous Focusing Channels

Take

$$\boxed{\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}}$$

- Real transport channels have s-varying focusing functions
- For a rough correspondence to physical lattices take: $k_{\beta 0} = \sigma_0 / L_p$

A valid family of equilibria can be constructed for any choice of function:

$$\boxed{f_\perp = f_\perp(H_\perp) \geq 0} \qquad H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

ϕ must be calculated consistently from the nonlinear Poisson equation:

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\epsilon_0}\int d^2x_\perp' \; f_\perp(H_\perp)}$$

- Solutions generated will be steady-state $(\partial/\partial s = 0)$
- It can be shown that the Poisson equation only has solutions with $(\partial/\partial\theta = 0)$

The Hamiltonian is only equivalent to the Courant-Snyder invariant in continuous focusing. In periodic focusing channels $\kappa_x(s)$ and $\kappa_y(s)$ vary in s and the Hamiltonian is not a constant of the motion.

The axisymmetric Poisson equation simplifies to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{qn}{\epsilon_0} = -\frac{q}{\epsilon_0}\int d^2x_\perp' \; f_\perp(H_\perp)$$

Introduce a streamfunction

$$\boxed{\psi(r) = \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}} \qquad r = \sqrt{x^2+y^2}$$

then

$$\boxed{H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \psi}$$

and system axisymmetry can be exploited to calculate the beam density as

$$n(r) = \int d^2x_\perp' \; f_\perp(H_\perp) \; = 2\pi\int_\psi^\infty dH_\perp \; f_\perp(H_\perp)$$

Then the Poisson equation can be recast in terms of the stream function as

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) = 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0\gamma_b^3\beta_b^2c^2}\int_{\psi(r)}^\infty dH_\perp \; f_\perp(H_\perp)}$$

To characterize a choice of equilibrium function $f_\perp(H_\perp)$, the (transformed) Poisson equation must be solved

- Equation is, in general, highly nonlinear rendering the procedure difficult

Some general features of equilibria can still be understood in terms of moments

- Apply rms equivalent beam picture

Moment properties of continuous focusing equilibrium distributions

Equilibria satisfy the rms equivalent matched beam envelope equation:

$$\boxed{k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0}$$

- Describes average radial force balance of particles

where $$\langle \cdots \rangle_\perp = \frac{\int d^2x_\perp \int d^2x'_\perp \ \cdots f_\perp(H_\perp)}{\int d^2x_\perp \int d^2x'_\perp \ f_\perp(H_\perp)}$$

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const} \qquad \lambda = q\int d^2x_\perp \int d^2x'_\perp \ f_\perp(H_\perp)$$

$$r_b^2 = 2\langle r^2\rangle_\perp = \frac{\int_0^\infty dr\ r^3 \int_\psi^\infty dH_\perp \ f_\perp(H_\perp)}{\int_0^\infty dr\ r \int_\psi^\infty dH_\perp \ f_\perp(H_\perp)}$$

$$\varepsilon^2 = 2r_b^2\langle \mathbf{x}'^2_\perp\rangle_\perp = 2r_b^2 \frac{\int_0^\infty dr\ r \int_\psi^\infty dH_\perp \ (H_\perp - \psi) f_\perp(H_\perp)}{\int_0^\infty dr\ r \int_\psi^\infty dH_\perp \ f_\perp(H_\perp)}$$

Parameters used to define

$$f_\perp(H_\perp)$$

should be cast in terms of

$$Q, \quad \varepsilon, \quad r_b$$

for use in accelerator applications. The rms equivalent beam equations can be used to carry out needed parameter eliminations. Such eliminations can be highly nontrivial due to the nonlinear form of the equations.

A kinetic temperature can also be calculated

$$T_x = \langle x'^2 \rangle_{\mathbf{x}'_\perp} \qquad\qquad \langle \cdots \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \ \cdots f_\perp}{\int d^2x'_\perp \ f_\perp}$$

$$\boxed{n(r)T_x(r) = \frac{1}{2}\int d^2x'_\perp \ \mathbf{x}'^2_\perp f_\perp(H_\perp) \ = 2\pi \int_\psi^\infty dH_\perp \ (H_\perp - \psi) f_\perp(H_\perp)}$$

which is also related to the emittance,

$$\langle x'^2 \rangle_\perp = \frac{\int d^2x_\perp \ nT_x}{\int d^2x_\perp n} \qquad\qquad \boxed{\varepsilon^2 = 16\langle x^2\rangle_\perp \langle x'^2 \rangle_\perp = 4r_b^2 \frac{\int d^2x_\perp \ nT}{\int d^2x_\perp \ n}}$$

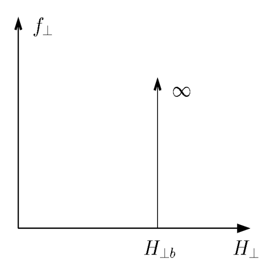
Choices of continuous focusing equilibrium distributions:

Common choices for $f_\perp(H_\perp)$ analyzed in the literature:

1) KV (already covered)

$$\boxed{f_\perp \propto \delta(H_\perp - H_{\perp b})}$$

$$H_{\perp b} = \text{const}$$

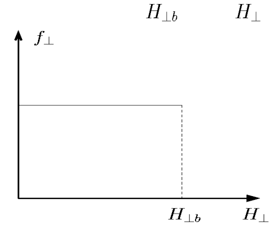


2) Waterbag (to be covered)

[see M. Reiser, *Charged Particle Beams*, (1994)]

$$\boxed{f_\perp \propto \Theta(H_{\perp b} - H_\perp)}$$

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x \end{cases}$$

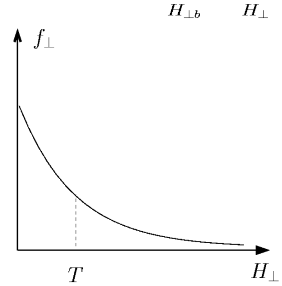


3) Thermal (to be covered)

[see M. Reiser; Davidson, *Noneutral Plasmas*, 1990]

$$\boxed{f_\perp \propto \exp(-H_\perp/T)}$$

$$T = \text{const} > 0$$



Infinity of choices can be made for an infinity of papers!

- Fortunately, range of behavior can be understood with a few reasonable choices

S6: Continuous Focusing: The Waterbag Equilibrium Distribution:

[see Reiser, Theory and Design of Charged Particle Beams, Wiley (1994)]

Waterbag distribution:

$$f_\perp(H_\perp) = f_0 \Theta(H_b - H_\perp)$$

$f_0 = \text{const}$

$H_b = \text{const}$ Edge Hamiltonian

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

The physical edge radius r_e of the beam will be related to the edge Hamiltonian:

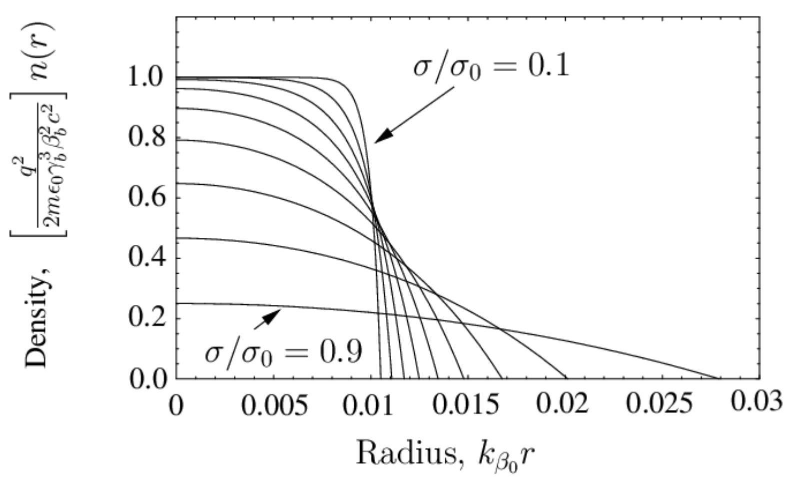
$$H_\perp|_{r=r_e} = H_b$$

Employing the general formulation, the Poisson equation for this choice can be analytically solved simplifying analysis.

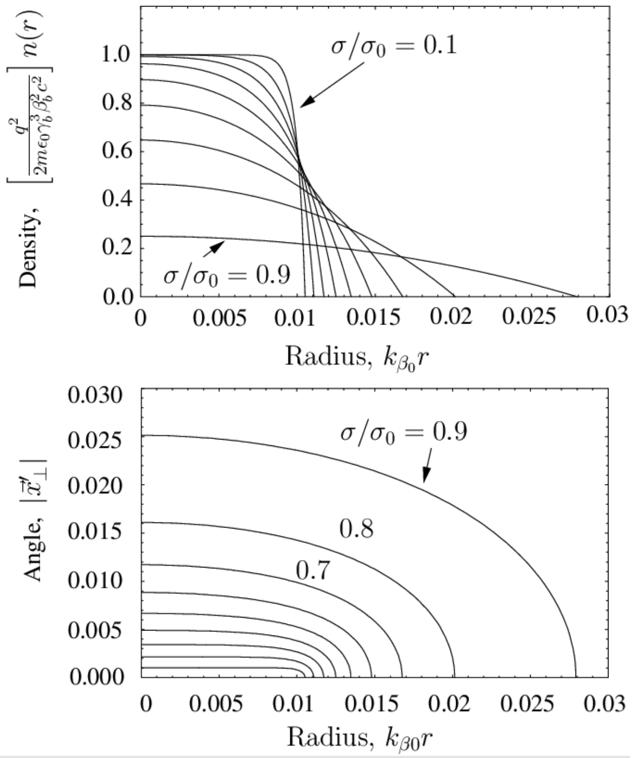
Details of Waterbag analysis to be included in later editions of notes.

1) Density profile at fixed line charge and focusing strength

$Q = 10^{-4}$ $k_{\beta 0}^2 = \text{const}$



2) Phase-space boundary at fixed line charge and focusing strength
$Q = 10^{-4}$ $k_{\beta 0}^2 = \text{const}$



Density Profile

Edge of distribution in phase-space

Scaled parameters for examples

σ/σ_0	s_b	$\frac{k_{\beta 0}^2 r_b^2}{Q}$	$k_0 r_e$	$\frac{r_e}{r_b}$	$Q = 10^{-4}$ $\frac{k_0}{k_{\beta 0}}$	$10^3 \times k_{\beta 0}\varepsilon_b$
0.9	0.2502	5.263	1.112	1.217	39.81	0.4737
0.8	0.4666	2.778	1.709	1.208	84.87	0.2222
0.7	0.6477	1.961	2.304	1.197	137.5	0.1373
0.6	0.7916	1.563	2.979	1.183	201.5	0.09375
0.5	0.8968	1.333	3.821	1.166	283.8	0.06667
0.4	0.9626	1.190	4.978	1.144	398.7	0.04762
0.3	0.9928	1.099	6.789	1.118	579.3	0.03297
0.2	0.9997	1.042	10.25	1.085	925.6	0.02083
0.1	1.0000	1.010	20.38	1.046	1938.	0.01010

S7: Continuous Focusing: The Thermal Equilibrium Distribution:

[see Davidson, Physics of Nonneutral Plasma, Addison Wesley (1990) and Reiser, Theory and Design of Charged Particle Beams, Wiley (1994)]

In an infinitely long continuous focusing channel, collisions will eventually relax the beam to thermal equilibrium. The Fokker-Planck equation predicts that the unique Maxwell-Boltzmann distribution describing this limit is:

$$\lim_{s\to\infty} f_\perp \propto \exp\left(-\frac{H_{\text{rest}}}{T}\right)$$

$H_{\text{rest}} =$ single particle Hamiltonian of beam in rest frame (energy units)

$T = \text{const}$ Thermodynamic temperature (energy units)

Beam propagation time in transport channel is generally short relative to collision time, inhibiting full relaxation

- Collective effects may enhance relaxation rate
  - Wave spectrums likely large for real beams and enhanced by transient and nonequilibrium effects
  - Random errors acting on system may enhance and lock-in phase mixing

Continuous focusing thermal equilibrium distribution

Analysis of the rest frame transformation shows that the 2D Maxwell-Boltzmann distribution (careful on frame for temperature definition!) is:

$$f_\perp(H_\perp) = \frac{m\gamma_b\beta_b^2c^2\hat{n}}{2\pi T}\exp\left(-\frac{m\gamma_b\beta_b^2c^2H_\perp}{T}\right)$$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$
$$= \frac{1}{2}\mathbf{x}_\perp'^2 + \psi$$

$T = \text{const}$ Temperature (energy units, lab frame)

$n(r=0) = \hat{n} = \text{const}$ on-axis density

$\phi(r=0) = 0$ (reference choice)

The density can then be conveniently calculated in terms of a scaled stream function:

$$n(r) = \int d^2x_\perp'\ f_\perp \quad = \hat{n}e^{-\tilde{\psi}}$$

$$\tilde{\psi}(r) \equiv \frac{m\gamma_b\beta_b^2c^2\psi}{T} = \frac{1}{T}\left(\frac{m\gamma_b\beta_b^2c^2k_{\beta 0}^2}{2}r^2 + \frac{q\phi}{\gamma_b^2}\right)$$

and the x- and y-temperatures are equal and spatially uniform with:

$$T_x = \gamma_b m\beta_b^2c^2\frac{\int d^2x_\perp'\ x'^2\ f_\perp}{\int d^2x_\perp'\ f_\perp} = T = \text{const}$$

Scaled Poisson equation for continuous focusing thermal equilibrium

To describe the thermal equilibrium density profile, the Poisson equation must be solved. In terms of the scaled streamfunction:

$$\boxed{\begin{array}{l} \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \tilde{\psi}}{\partial \rho}\right) = 1 + \Delta - e^{-\tilde{\psi}} \\ \tilde{\psi}(\rho = 0) = 0 \qquad \frac{\partial \tilde{\psi}}{\partial \rho}(\rho = 0) = 0 \end{array}}$$

Here,

$$\lambda_D = \left(\frac{\epsilon_0 T}{q^2 \hat{n}}\right)^{1/2}$$
Debye length formed from the peak, on-axis beam density

$$\rho = \frac{r}{\gamma_b \lambda_D}$$
Scaled radial coordinate in rel. Debye lengths

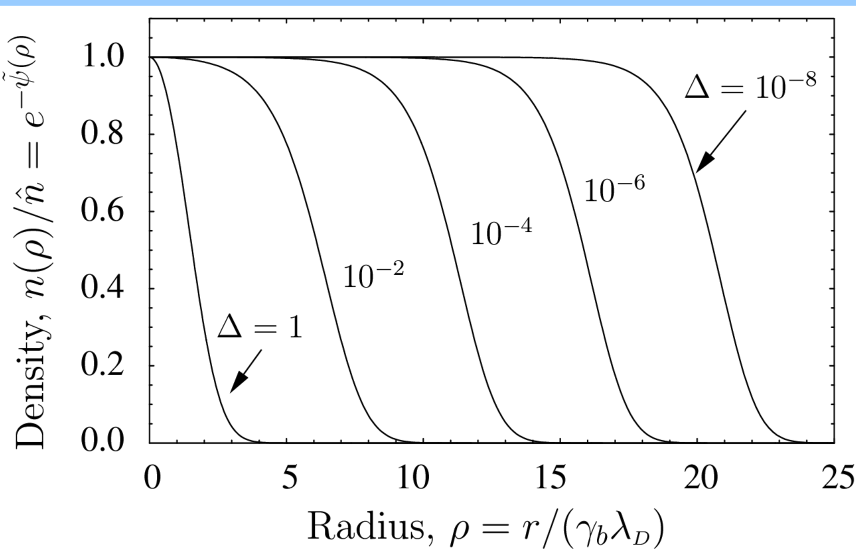
$$\hat{\omega}_p \equiv \left(\frac{q^2 \hat{n}}{\epsilon_0 m}\right)^{1/2}$$
Plasma frequency formed from on-axis beam density

$$\longrightarrow \qquad \lambda_D = \left(\frac{T}{\hat{\omega}_p^2 m}\right)^{1/2}$$

$$\Delta = \frac{2\gamma_b^3 \beta_b^2 c^2 k_{\beta 0}^2}{\hat{\omega}_p^2} - 1$$
Dimensionless parameter relating the ratio of applied to space-charge defocusing forces

- Equation is highly nonlinear and must, in general, be solved numerically
- Scaled solutions depend only on the single dimensionless parameter Δ

Numerical solution of scaled thermal equilibrium Poisson equation in terms of a normalized density



- Equation is highly nonlinear and must, in general, be solved numerically
  - Dependance on Δ is very sensitive
  - For small Δ, the beam is nearly uniform in the core
- Edge fall-off is always in a few Debye lengths when Δ is small
  - Edge becomes very sharp at fixed beam line-charge

Parameters constraints for the thermal equilibrium beam

Parameters employed in $f_\perp(H_\perp)$ to specify the equilibrium are (+ kinematic factors):

$$\hat{n}, \quad T, \quad \Delta$$

Parameters preferred for accelerator applications:

$$k_{\beta 0}, \quad Q, \quad \varepsilon_x = \varepsilon_y = \varepsilon_b$$

Needed constraints can be calculated directly from the equilibrium:

$$\boxed{\begin{aligned} Q &= \left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) \int_0^\infty d\rho\rho\, e^{-\tilde{\psi}} \\ k_{\beta 0}^2 \varepsilon_b &= 4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right)\left[4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) + Q\right] \\ k_{\beta 0}^2 &= \left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) \frac{1+\Delta}{2(\gamma_b \lambda_D)^2} \end{aligned}}$$

Integral function of Δ only

Also useful,

$$\boxed{r_b^2 = 4\langle x^2\rangle_\perp = \frac{1}{k_{\beta 0}^2}\left[4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) + Q\right]}$$

These constraints must, in general, be solved numerically

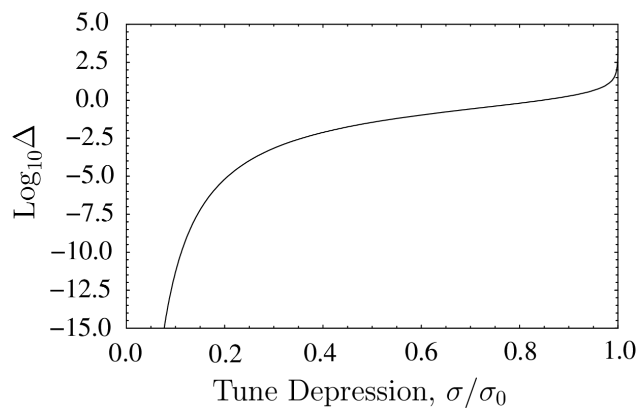
- Useful to probe system sensitivities in relevant parameters

Examples:

1) rms equivalent beam tune depression as a function of Δ

$$\frac{\sigma}{\sigma_0} = \sqrt{1 - \frac{Q}{k_{\beta_0}^2 r_b^2}} = \left\{1 - \frac{[\int_0^\infty d\rho\, \rho e^{-\tilde{\psi}}]^2}{(1+\Delta)\int_0^\infty d\rho \rho^3 e^{-\tilde{\psi}}}\right\}^{1/2}$$

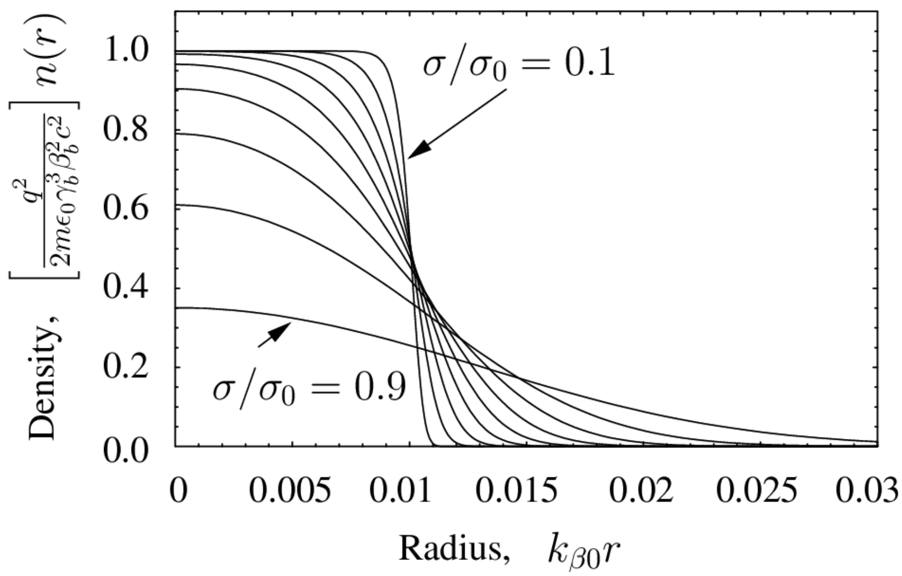
R.H.S function of Δ only



- Small tune depression corresponds to *extremely* small values of Δ
  - Special numerical methods must be employed to calculate

2) Density profile at fixed line charge and focusing strength

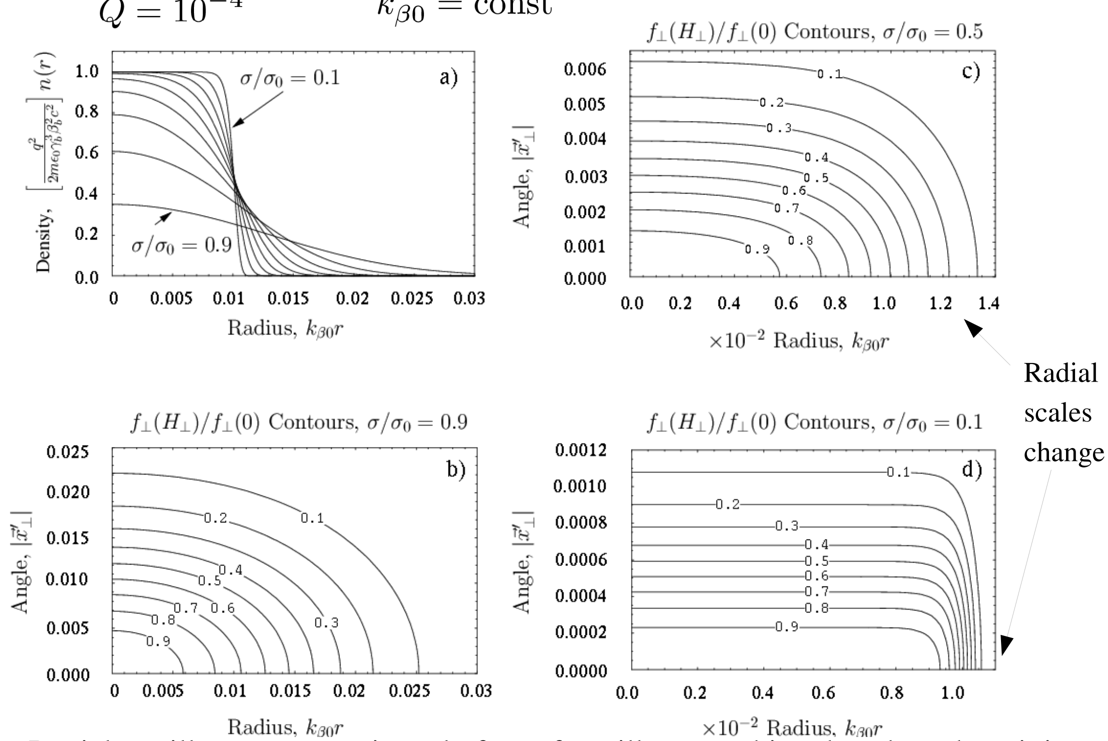
$Q = 10^{-4}$ $\qquad$ $k_{\beta 0}^2 = \text{const}$



- Density profile changes with scaled T
  - Low values yields a flat-top => $\sigma/\sigma_0 \to 0$
  - High values yield a Gaussian like profile => $\sigma/\sigma_0 \to 1$

3) Distribution contours at fixed line charge and focusing strength

$Q = 10^{-4}$ $\qquad$ $k_{\beta 0}^2 = \text{const}$



- Particles will move approximately force-free till approaching the edge where it is rapidly bent back (see Debye screening analysis this lecture)

Scaled parameters for examples 2) and 3)

				$Q = 10^{-4}$	
σ/σ_0	Δ	s_b	$k_{\beta 0}\gamma_b\lambda_D$	$\frac{T}{m\gamma_b\beta_b^2c^2}$	$10^3 \times k_{\beta 0}\varepsilon_b$
0.9	1.851	0.3508	12.33	1.065×10^{-4}	0.4737
0.8	6.382×10^{-1}	0.6104	6.034	4.444×10^{-5}	0.2222
0.7	2.649×10^{-1}	0.7906	3.898	2.402×10^{-5}	0.1373
0.6	1.059×10^{-1}	0.9043	2.788	1.406×10^{-5}	0.09375
0.5	3.501×10^{-2}	0.9662	2.077	8.333×10^{-6}	0.06667
0.4	7.684×10^{-3}	0.9924	1.549	4.762×10^{-6}	0.04762
0.3	6.950×10^{-4}	0.9993	1.112	2.473×10^{-6}	0.03297
0.2	6.389×10^{-6}	1.0000	0.7217	1.042×10^{-6}	0.02083
0.1	4.975×10^{-12}	1.0000	0.3553	2.525×10^{-7}	0.01010

Comments on continuous focusing thermal equilibria

From these results it is not surprising that the KV model works well for real beams with strong space-charge (i.e, rms equivalent σ/σ_0 small) since the edges of a smooth thermal distribution become sharp

- Thermal equilibrium likely overestimates the edge with since T = const, whereas a real distribution likely becomes colder near the edge

However, the beam edge contains strong nonlinear terms that will cause deviations from the KV model

- Nonlinear terms can radically change the stability properties (stabilize fictitious higher order KV modes)
- Smooth distributions contain a spectrum of particle oscillation frequencies that are amplitude dependent

S8: Continuous Focusing: Debye Screening in a Thermal Equilibrium Beam

[Davidson, Physics of Nonneutral Plasmas, Addison Wesley (1990)]

We will show that space-charge and the applied focusing forces of the lattice conspire together to Debye screen interactions in the core of a beam with high space-charge intensity

- Will systematically derive the Debye length employed in the intro lectures of J.J. Barnard
- The applied focusing forces are analogous to a stationary neutralizing species in a plasma

// Review:

Free-space field of a "bare" test line-charge λ_t at the origin $r = 0$

$$\rho(r) = \lambda_t \frac{\delta(r)}{2\pi r} \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = -\frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

solution (use Gauss' theorem) shows long-range interaction

$$\boxed{\begin{aligned} \phi &= -\frac{\lambda_t}{2\pi\epsilon_0}\ln(r) + \text{const} \\ E_r &= -\frac{\partial \phi}{\partial r} = \frac{\lambda_t}{2\pi\epsilon_0 r} \end{aligned}}$$

//

Place a *small* test line charge at *r* = 0 in a thermal equilibrium beam:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; f_\perp(H_\perp) \; - \; \frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

Thermal Equilibrium Test Line-Charge

Set:

$$\phi = \phi_0 + \delta\phi$$

$\phi_0 =$ Thermal Equilibrium potential with no test line-charge

$\delta\phi =$ Perturbed potential from test line-charge

Assume thermal equilibrium adapts adiabatically to the test line-charge:

$$n(r) = \int d^2x'_\perp \; f_\perp(H_\perp) \; = \hat{n}e^{-\tilde{\psi}} \; \simeq \hat{n}e^{-\tilde{\psi}_0(r)}e^{-q\delta\phi/(\gamma_b^2 T)}$$

$$\simeq \hat{n}e^{-\tilde{\psi}_0(r)}\left(1 - \frac{q\delta\phi}{\gamma_b^2 T}\right) \qquad \left|\frac{q\delta\phi}{\gamma_b^2 T}\right| \ll 1$$

Yields:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) = -\frac{q^2}{\epsilon_0\gamma_b^2 T}\hat{n}e^{-\tilde{\psi}_0(r)} \; - \; \frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

Assume a relatively cold beam so the density is flat near the test line-charge:

$$\hat{n}e^{-\tilde{\psi}_0(r)} \simeq \hat{n}$$

This gives:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) - \frac{\delta\phi}{\gamma_b^2\lambda_D^2} = -\frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}}$$

$$\lambda_D = \left(\frac{\epsilon_0 T}{q^2\hat{n}}\right)^{1/2} = \text{Debye radius formed from peak, on-axis beam density}$$

Derive a general solution by connecting solution very near the test charge with the general solution for r nonzero:

Near solution: $(r \rightarrow 0)$

$$\frac{\delta\phi}{\gamma_b^2\lambda_D^2} \quad \text{Negligible --->} \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) = -\frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

The free-space solution can be immediately applied:

$$\boxed{\delta\phi \simeq -\frac{\lambda_t}{2\pi\epsilon_0}\ln(r) + \text{const} \qquad r \rightarrow 0}$$

General Exterior Solution: $(r \neq 0)$

The delta-function term vanishes giving:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \delta\phi}{\partial \rho}\right) - \delta\phi = 0 \qquad \rho \equiv \frac{r}{\gamma_b\lambda_D}$$

This is a modified Bessel equation of order 0 with general solution:

$$\delta\phi = C_1 I_0(\rho) + C_2 K_0(\rho)$$

$I_0(x) =$ Modified Bessel Func, 1st kind

$K_0(x) =$ Modified Bessel Func, 2nd kind

$C_1, \;\; C_2 =$ constants

Connection and General Solution:

Use limiting forms:

$\rho \ll 1$

$$I_0(\rho) \rightarrow 1 + \Theta(\rho^2)$$

$$K_0(\rho) \rightarrow -[\ln(\rho/2) + 0.5772\cdots + \Theta(\rho^2)]$$

$\rho \gg 1$

$$I_0(\rho) \rightarrow \frac{e^\rho}{\sqrt{2\pi\rho}}[1 + \Theta(1/\rho)]$$

$$K_0(\rho) \rightarrow \sqrt{\frac{\pi}{2\rho}}[1 + \Theta(1/\rho)]$$

Comparison shows that we must choose for connection to the near solution and regularity at infinity:

$$\boxed{\begin{aligned} C_1 &= 0 \\ C_2 &= \frac{\lambda_t}{2\pi\epsilon_0} \end{aligned}}$$

General solution shows Debye screening of test charge in the core of the beam:

$$\begin{aligned} \delta\phi &= \frac{\lambda_t}{2\pi\epsilon_0} K_0\left(\frac{r}{\gamma_b\lambda_D}\right) \\ &\simeq \frac{\lambda_t}{2\sqrt{2\pi}\epsilon_0}\frac{1}{\sqrt{r/(\gamma_b\lambda_D)}} e^{-r/(\gamma_b\lambda_D)} \qquad r \gg \gamma_b\lambda_D \end{aligned}$$

$K_0(x)$ Order Zero Modified Bessel Function

- Screened interaction does not require overall charge neutrality!
  - Beam particles redistribute to screen bare interaction
  - Beam behaves as a plasma and expect similar collective waves etc.
- Same result for all smooth equilibrium distributions and in 1D, 2D, and 3D
  - Reason why lower dimension models can get the "right" answer for collective interactions in spite of the Coulomb force varying with dimension
- Explains why the radial density profile in the core of space-charge dominated beams are expected to be flat

S9: Continuous Focusing: The Density Inversion Theorem
Shows x and x' dependancies are strongly connected in an equilibrium

For:

$$f_\perp = f_\perp(H_\perp)$$

$$\begin{aligned} H_\perp &= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \\ &= \frac{1}{2}\mathbf{x}_\perp'^2 + \psi(r) \end{aligned} \qquad \psi \equiv \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

calculate the beam density

$$n(r) = \int d^2x'_\perp\; f_\perp(H_\perp) = 2\pi\int_0^\infty dU\; f_\perp(U+\psi(r))$$

differentiate:

$$\begin{aligned} \frac{\partial n}{\partial\psi} &= 2\pi\int_0^\infty dU\;\frac{\partial}{\partial\psi}f_\perp(U+\psi) = 2\pi\int_0^\infty dU\;\frac{\partial}{\partial U}f_\perp(U+\psi) \\ &= 2\pi\underbrace{\lim_{U\to\infty} f_\perp(U+\psi)}_{0} - 2\pi f_\perp(\psi) \end{aligned}$$

bounded distribution

$$\boxed{f_\perp(H_\perp) = -\frac{1}{2\pi}\frac{\partial n}{\partial\psi}\bigg|_{\psi=H_\perp} \qquad \psi(r) = \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi(r)}{m\gamma_b^3\beta_b^2c^2}}$$

Assume that $n(r)$ is specified, then the Poisson equation can be integrated:

$$\psi(r) - \frac{q\phi(r=0)}{m\gamma_b^3\beta_b^2c^2} = \frac{1}{2}k_{\beta 0}^2 r^2 - \frac{q}{m\gamma_b^3\beta_b^2c^2\epsilon_0}\int_0^r\frac{d\tilde{r}}{\tilde{r}}\int_0^{\tilde{r}} d\tilde{\tilde{r}}\;\tilde{\tilde{r}}\;n(\tilde{\tilde{r}})$$

For $n(r) = \text{const}$
$$\int_0^r \frac{d\tilde{r}}{\tilde{r}} \int_0^{\tilde{r}} d\tilde{\tilde{r}}\, \tilde{\tilde{r}}\, n(\tilde{\tilde{r}}) \quad \propto r^2$$

This suggests that $\psi(r)$ is monotonic in r when $d\, n(r)/dr$ is monotonic. Apply the chain rule:

Density Inversion Theorem

$$\boxed{\begin{aligned} f_\perp(H_\perp) &= -\frac{1}{2\pi}\frac{\partial n}{\partial \psi}\bigg|_{\psi=H_\perp} = -\frac{1}{2\pi}\frac{\partial n(r)/\partial r}{\partial \psi(r)/\partial r}\bigg|_{\psi=H_\perp} \\ \psi(r) &= \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^2} \end{aligned}}$$

For specified monotonic $n(r)$ the density inversion theorem can be applied with the Poisson equation to calculate the corresponding equilibrium $f_\perp(H_\perp)$

Comments on density inversion theorem:

- Shows that the x and x' dependance of the distribution are *inextricably linked* for an equilibrium distribution function $f_\perp(H_\perp)$
  - Not so surprising -- equilibria are highly constrained
- If $df_\perp(H_\perp)/dH_\perp \leq 0$ then the kinetic stability theorem (see Kinetic Stability lectures) shows that the equilibrium is also stable

// Example: Application of the inversion theorem to the KV equilibrium

$$n = \begin{cases} \hat{n}, & 0 \leq r < r_b \\ 0, & r_b < r \end{cases} \longrightarrow \frac{\partial n}{\partial r} = -\hat{n}\delta(r - r_b)$$

$$\begin{aligned} \frac{\partial n}{\partial \psi} &= \frac{\partial n/\partial r}{\partial \psi/\partial r} \\ &= -\frac{\hat{n}\delta(r-r_b)}{\partial\psi/\partial r} \\ &= -\frac{\hat{n}\delta(r-r_b)}{\partial\psi/\partial r|_{r=r_b}} \\ &= -\hat{n}\delta(\psi(r) - \psi(r_b)) \end{aligned}$$

property of delta-function:

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|df/dx|_{x=x_i}}$$

$$f(x_i) = 0$$

x_i root of f(x)

use: $\psi(r_b) = H_\perp|_{\mathbf{x}'_\perp = 0} = H_{\perp b}$

$$\longrightarrow \boxed{f_\perp(H_\perp) = -\frac{1}{2\pi}\frac{\partial n}{\partial \psi}\bigg|_{\psi=H_\perp} = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})}$$

Expected KV form //

Similar application of derivatives with respect to Courant-Snyder invariants can "derive" the needed form for the KV distribution of an elliptical beam without guessing.

S10: Comments on the plausibility of smooth, Vlasov equilibria in periodic transport channels

The KV and continuous models are the only (or related to simple transforms thereof) known exact beam equilibria. Both suffer from idealizations that render them inappropriate for use as initial distribution functions for modeling of real accelerator systems:

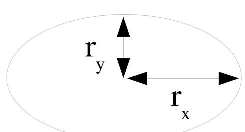
- KV distribution has an unphysical structure giving rise to well known instabilities with unphysical manifestations
- Continuous focusing is inadequate to model real accelerator lattices with periodic or s-varying focusing forces

There is much room for improvement in this area, including study if smooth equilibria exist in periodic focusing and implications if no exact equilibria exist.

Large envelope flutter associated with strong focusing can result in a rapid high-order oscillating force imbalance acting on edge particles of the beam

Temperature Flutter

Elliptical rms Equivalent Beam



$$\varepsilon_x^2 \propto T_x r_x^2 \simeq \text{const} \implies T_x \propto \frac{1}{r_x^2}$$

Example Systems	$(r_{max}/r_{min})^2$
AG Trans: $\sigma_0 = 60^\circ$	~ 2.5
AG Trans: $\sigma_0 = 100^\circ$	~ 4.9
Matching Section	~ 15 Possible

Characteristic Plasma Frequency of Collective Effects

Continuous Focusing Estimate

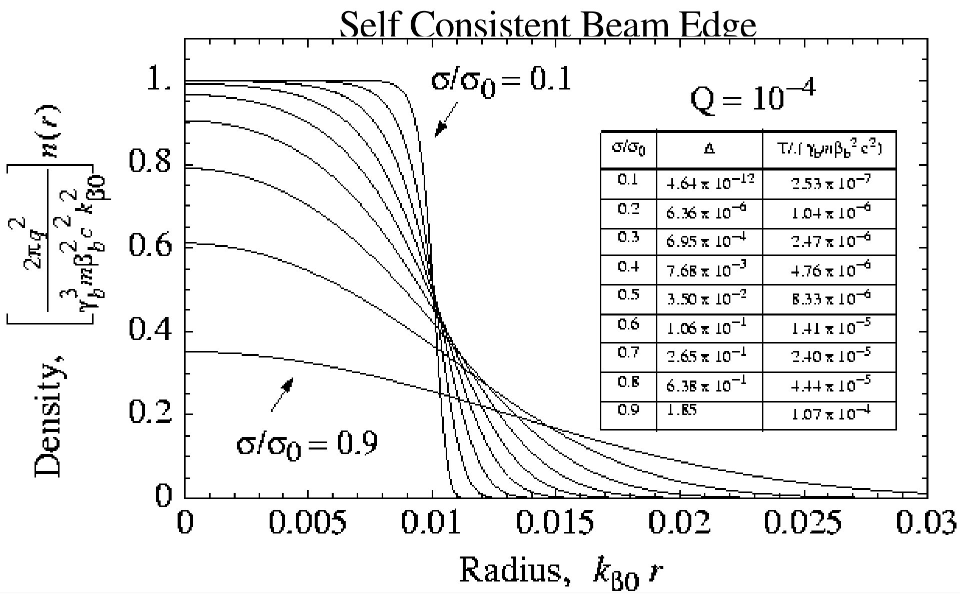
$$\sigma_{\text{plasma}} \sim \frac{L_p}{r_b}\sqrt{2Q} \qquad \text{Typical: } \sigma_{\text{plasma}} \sim 105^\circ/\text{period}$$

- Temperature asymmetry in beam will rapidly fluctuate with lattice periodicity
  - Converging plane => Warmer
  - Diverging plane => Colder
- Collective plasma wave response slower than lattice frequency
  - Beam edge will not be able to adapt rapidly enough
  - Collective waves will be launched from lack of local force balance near the edge

The continuous focusing equilibrium distribution suggests that varying Debye screening together with envelope flutter would require a rapidly adapting beam edge in a smooth, periodic equilibrium beam distribution

$$f_\perp = \frac{m\gamma_b\beta_b^2c^2\hat{n}}{2\pi T}\exp\left(-\frac{m\gamma_b\beta_b^2c^2H_\perp}{T}\right)$$

Continuous Focusing Thermal Equilibrium Beam



σ/σ_0	Δ	$T/(\gamma_b m\beta_b^2 c^2)$
0.1	4.64×10^{-12}	2.53×10^{-7}
0.2	6.36×10^{-6}	1.04×10^{-6}
0.3	6.95×10^{-4}	2.47×10^{-6}
0.4	7.68×10^{-3}	4.76×10^{-6}
0.5	3.50×10^{-2}	8.33×10^{-6}
0.6	1.06×10^{-1}	1.41×10^{-5}
0.7	2.65×10^{-1}	2.40×10^{-5}
0.8	6.38×10^{-1}	4.44×10^{-5}
0.9	1.85	1.07×10^{-4}

It is clear from these considerations that if smooth "equilibrium" beam distributions exist for periodic focusing, then they are highly nontrivial

Would a nonexistence of an equilibrium distribution be a problem:

- Real beams are born off a source that can be simulated
  - Propagation length can be relatively small in linacs
- Transverse confinement can exist without an equilibrium
  - Particles can turn at large enough radii forming an edge
  - Edge can oscillate from lattice period to lattice period without pumping to large excursions

$\longrightarrow$ Might not preclude long propagation with preserved statistical beam quality

Even approximate equilibria would help sort out complicated processes:

- Reduce transients and fluctuations can help understand processes in simplest form
  - Allows more "plasma physics" type analysis and advances
- Beams in Vlasov simulations are often observed to "settle down" to a fairly regular state after an initial transient evolution
  - Extreme phase mixing leads to an effective relaxation

These slides will be corrected and expanded for reference and any future editions of the US Particle Accelerator School class:
Beam Physics with Intense Space Charge, by J.J. Barnard and S.M. Lund

Corrections and suggestions are welcome. Contact:

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Appendix A

Self Fields of a Uniform Density Elliptical Beam in Free Space

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = \begin{cases} \dfrac{-\lambda}{\pi\varepsilon_0 r_x r_y}; & \dfrac{x^2}{r_x^2}+\dfrac{y^2}{r_y^2} < 1 \\ 0; & \dfrac{x^2}{r_x^2}+\dfrac{y^2}{r_y^2} > 1 \end{cases}$$

$$\frac{\partial\phi}{\partial r} \sim \frac{\lambda}{2\pi\varepsilon_0 r} \quad \text{as} \quad r \to \infty.$$

The solution to this system to an arb. constant has been formally constructed by Landau & Lifshitz and others (gravitational field analog) as:

$$\phi = \frac{-\lambda}{4\pi\varepsilon_0}\left\{\int_0^{\xi}\frac{ds}{\left[(r_x^2+s)(r_y^2+s)\right]^{1/2}} + \int_{\xi}^{\infty}\frac{ds}{\left[(r_x^2+s)(r_y^2+s)\right]^{1/2}}\left(\frac{x^2}{r_x^2+s}+\frac{y^2}{r_y^2+s}\right)\right\} + \text{const.}$$

where

$$\begin{cases} \xi = 0; & \text{when } (x/r_x)^2+(y/r_y)^2 < 1 \\ \xi: \text{ root of } \dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi} = 1, & \text{when } (x/r_x)^2+(y/r_y)^2 > 1 \end{cases}$$

Trivially for $x=y=0$

$$\phi(x=y=0) = \text{const.}$$

Calculate:

$$\frac{\partial\phi}{\partial x} = \frac{-\lambda}{4\pi\varepsilon_0}\left\{\int_{\xi}^{\infty}\frac{ds}{\left[(r_x^2+s)(r_y^2+s)\right]^{1/2}}\frac{2x}{r_x^2+s} - \frac{1}{\left[(r_x^2+\xi)(r_y^2+\xi)\right]^{1/2}}\left[1-\frac{x^2}{r_x^2+\xi}-\frac{y^2}{r_y^2+\xi}\right]\frac{\partial\xi}{\partial x}\right\}$$

$$\left.\begin{array}{l} \text{If } \xi \neq 0 \Rightarrow 1 = \dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi} \\ \xi = 0 \Rightarrow \dfrac{\partial\xi}{\partial x} = 0 \end{array}\right\} \Rightarrow \text{2nd term vanishes}$$

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$$\boxed{\begin{aligned}\frac{\partial\phi}{\partial x} &= -\frac{\lambda}{2\pi\varepsilon_0}\int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\frac{x}{r_x^2+s}\\ &\text{by symmetry}\\ \frac{\partial\phi}{\partial y} &= -\frac{\lambda}{2\pi\varepsilon_0}\int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\frac{y}{r_y^2+s}\end{aligned}}$$

Differentiating again and using the chain rule:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\varepsilon_0}\left\{\int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\left[\frac{1}{r_x^2+s}+\frac{1}{r_y^2+s}\right] - \frac{1}{[(r_x^2+\xi)(r_y^2+\xi)]^{1/2}}\left[\frac{x\,\partial\xi/\partial x}{r_x^2+\xi}+\frac{y\,\partial\xi/\partial y}{r_y^2+\xi}\right]\right\}$$

Must show that the r.h.s. reduces to the needed forms for:

case 1 exterior ξ satisfies: $\dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi}=1$

case 2 interior $\xi = 0$

case 1 (exterior: $x^2/r_x^2 + y^2/r_y^2 > 1$)

Differentiate $\dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi}=1$

$$\Rightarrow \quad \frac{\partial\xi}{\partial x} = \frac{2x}{(r_x^2+\xi)\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]}$$

$$\frac{\partial\xi}{\partial y} = \frac{2y}{(r_y^2+\xi)\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]}$$

$$\Rightarrow \frac{x\,\partial\xi/\partial x}{r_x^2+\xi}+\frac{y\,\partial\xi/\partial y}{r_y^2+\xi} = 2\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]\frac{1}{\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]} = 2$$

Also need Integrals like: $\quad W^2 \equiv s + r_y^2$

$$I_x(\xi) \equiv \int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\frac{1}{r_x^2+s} = 2\int_{\sqrt{r_x^2+\xi}}^\infty \frac{dW}{(r_x^2-r_y^2+W^2)^{3/2}}$$

This integral can be done using tables:

$$I_x(\xi) = \left.\frac{2w}{(r_x^2-r_y^2)\sqrt{r_x^2-r_y^2+w^2}}\right|_{w=\sqrt{r_x^2+\xi}}^{w\to\infty} = \frac{2}{r_x^2-r_y^2} - \frac{2\sqrt{r_y^2+\xi}}{(r_x^2-r_y^2)\sqrt{r_x^2+\xi}}$$

Similarly:

$$I_y(\xi) = \int_\xi^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\frac{1}{(r_y^2+s)} = \frac{2}{r_y^2-r_x^2} - \frac{2\sqrt{r_x^2+\xi}}{(r_y^2-r_x^2)\sqrt{r_y^2+\xi}}$$

$$\int_0^\infty \frac{ds}{[(r_x^2+s)(r_y^2+s)]^{1/2}}\left[\frac{1}{r_x^2+s}+\frac{1}{r_y^2+s}\right] = I_x(\xi)+I_y(\xi)$$

$$= \frac{2}{r_x^2-r_y^2}\left(\frac{\sqrt{r_x^2+\xi}}{\sqrt{r_y^2+\xi}} - \frac{\sqrt{r_y^2+\xi}}{\sqrt{r_x^2+\xi}}\right) = \frac{2}{[(r_x^2+\xi)(r_y^2+\xi)]^{1/2}}$$

Using these results:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\varepsilon_0}\left\{\frac{2}{[(r_x^2+\xi)(r_y^2+\xi)]^{1/2}} - \frac{2}{[(r_x^2+\xi)(r_y^2+\xi)]^{1/2}}\right\}$$

$$= 0 \quad \text{checks.} \checkmark$$

Case 2 (Interior: $x^2/r_x^2 + y^2/r_y^2 < 1$)

$$\xi = 0 \Rightarrow \frac{x\,\partial\xi/\partial x}{r_x^2+\xi} + \frac{y\,\partial\xi/\partial y}{r_y^2+\xi} = 0$$

$$\Rightarrow I_x(\xi=0) = \cancel{I_y(\xi=0)} = \frac{2}{(r_x+r_y)r_x} \quad \text{and} \quad I_y(\xi=0) = \frac{2}{(r_x+r_y)r_y}$$

Using these results:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\varepsilon_0}\left\{\frac{2}{r_xr_y} - 0\right\} = -\frac{\lambda}{\varepsilon_0\pi r_x r_y} = -\frac{q\hat{n}}{\varepsilon_0} \quad \text{checks} \checkmark$$

Finally, check the limiting form outside the beam for r large $\Rightarrow$ ξ large.

$$-\frac{\partial\phi}{\partial x} = \frac{\lambda}{2\pi\varepsilon_0}x I_x(\xi) \qquad \lim_{r\to\infty} I_x(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$

$$-\frac{\partial\phi}{\partial y} = \frac{\lambda}{2\pi\varepsilon_0}y I_y(\xi) \qquad \lim_{r\to\infty} I_y(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$

Thus:

$$\lim_{r\to\infty} -\frac{\partial\phi}{\partial x} = \frac{\lambda}{2\pi\epsilon_0}\frac{x}{r^2} \checkmark$$

$$\lim_{r\to\infty} -\frac{\partial\phi}{\partial y} = \frac{\lambda}{2\pi\epsilon_0}\frac{y}{r^2} \checkmark$$

These have the correct limiting forms for a line charge at the origin. Completing the verification of the general formula.

In the beam $\left(x^2/r_x^2 + y^2/r_y^2 < 1,\ \xi = 0\right)$, the formula reduces to:

$$\phi = -\frac{\lambda}{4\pi\epsilon_0}\left\{x^2 I_x(\xi=0) + y^2 I_y(\xi=0)\right\} + \text{const.}$$

$$= -\frac{\lambda}{4\pi\epsilon_0}\left\{\frac{2x^2}{r_x(r_x+r_y)} + \frac{2y^2}{r_y(r_x+r_y)}\right\} + \text{const.}$$

$$\boxed{\phi = -\frac{\lambda}{2\pi\epsilon_0}\left\{\frac{x^2}{r_x(r_x+r_y)} + \frac{y^2}{r_y(r_x+r_y)}\right\} + \text{const.}}$$

The case of an axisymmetric beam with

$$r_x = r_y = r_b$$

is easy to construct explicitly and is included in the homework problems.

There is also an alternative way to do this field calculation, that is less direct but more efficient. We carry out this proof now since steps involved are useful for other purposes.

A density profile with elliptic symmetry can be expressed as:

$$n(x,y) = n\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$$

Here we do not assume a specific uniform density profile and we leave $n(x^2/r_x^2 + y^2/r_y^2)$ arbitrary outside of having elliptic symmetry. The solution to the 2D Poisson equation in free-space

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q n}{\varepsilon_0}$$

is then given by

$$\phi = -\frac{q r_x r_y}{4\varepsilon_0}\int_0^\infty d\xi \, \frac{\eta(\mathcal{X})}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

$$\mathcal{X} \equiv \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi}$$

where $\eta(\mathcal{X})$ is a function defined such that:

$$n(x,y) = \left.\frac{d\eta(\mathcal{X})}{d\mathcal{X}}\right|_{\xi=0}$$

This choice for $\eta(\mathcal{X})$ can always be made.

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We first prove that this solution is valid by direct substitution:

$$\mathcal{X} = \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} \quad \Rightarrow \quad \frac{\partial \mathcal{X}}{\partial x} = \frac{2x}{r_x^2+\xi} ; \quad \frac{\partial^2 \mathcal{X}}{\partial x^2} = \frac{2}{r_x^2+\xi}$$

$$\frac{\partial \mathcal{X}}{\partial y} = \frac{2y}{r_y^2+\xi} ; \quad \frac{\partial^2 \mathcal{X}}{\partial y^2} = \frac{2}{r_y^2+\xi}$$

Substitute in Poisson's equation and use the chain rule and results above:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q r_x r_y}{4\epsilon_0}\int_0^\infty d\xi \, \frac{\left(\frac{d^2\eta}{d\mathcal{X}^2}\right)\left(\frac{4x^2}{(r_x^2+\xi)^2} + \frac{4y^2}{(r_y^2+\xi)^2}\right) + \left(\frac{d\eta}{d\mathcal{X}}\right)\left(\frac{2}{r_x^2+\xi} + \frac{2}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

Note: $\quad d\mathcal{X} = -\left[\frac{x^2}{(r_x^2+\xi)^2} + \frac{y^2}{(r_y^2+\xi)}\right] d\xi$

so the first integral can be simplified by partial integration:

$$\int_0^\infty d\xi \, \frac{\left(\frac{d^2\eta}{d\mathcal{X}^2}\right)\left(\frac{4x^2}{(r_x^2+\xi)^2} + \frac{4y^2}{(r_y^2+\xi)^2}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}} = -4\int_0^\infty d\xi \, \frac{\frac{d^2\eta}{d\mathcal{X}^2}\frac{d\mathcal{X}}{d\xi}}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

$$= -4\int_0^\infty d\xi \, \frac{\frac{d}{d\xi}\left(\frac{d\eta}{d\mathcal{X}}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}} = -4\int_0^\infty d\xi \, \frac{d}{d\xi}\left[\frac{\frac{d\eta}{d\mathcal{X}}}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}\right] + 4\int_0^\infty d\xi \, \frac{d\eta}{d\mathcal{X}}\frac{d}{d\xi}\frac{1}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

$$= \left.\frac{-4\frac{d\eta}{d\mathcal{X}}}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}\right|_{\xi\to 0}^{\xi\to\infty} - 2\int_0^\infty d\xi \, \frac{\frac{d\eta}{d\mathcal{X}}\left(\frac{1}{r_x^2+\xi} + \frac{1}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

$$= \frac{4}{r_x r_y}\left.\frac{d\eta}{d\mathcal{X}}\right|_{\xi=0} - 2\int_0^\infty d\xi \, \frac{\frac{d\eta}{d\mathcal{X}}\left(\frac{1}{r_x^2+\xi} + \frac{1}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\,\sqrt{r_y^2+\xi}}$$

term will cancel 2nd Integral

Thus:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q \cancel{r_x r_y}}{\cancel{4}\varepsilon_0}\frac{\cancel{4}}{\cancel{r_x r_y}}\left.\frac{d\eta(\mathcal{X})}{d\mathcal{X}}\right|_{\xi=0}$$

But $\left.\frac{d\eta(\mathcal{X})}{d\mathcal{X}}\right|_{\xi=0} = n(x,y)$ by definition.

$$\Rightarrow \underline{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q\, n(x,y)}{\varepsilon_0}}$$ verifying the result.

For a uniform density ellipse we take:

$$\eta(\mathcal{X}) = \frac{\lambda}{q\pi r_x r_y}\begin{cases}\mathcal{X} & ; \mathcal{X}<1\\ 1 & ; \mathcal{X}>1\end{cases} \Rightarrow \frac{d\eta(\mathcal{X})}{d\mathcal{X}} = \begin{cases}\frac{\lambda}{q\pi r_x r_y} & ; \mathcal{X}<1\\ 0 & ; \mathcal{X}>1\end{cases}$$

Thus

$$\left.\frac{d\eta(\mathcal{X})}{d\mathcal{X}}\right|_{\xi=0} = \begin{cases}\frac{\lambda}{q\pi r_x r_y} & ; \left.\mathcal{X}\right|_{\xi=0}<1\\ 0 & ; \left.\mathcal{X}\right|_{\xi=0}\end{cases} = \begin{cases}\frac{\lambda}{q\pi r_x r_y} & ; \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}<1\\ 0 & ; \frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}>1\end{cases}$$

$\therefore$ $\underline{\left.\frac{d\eta(\mathcal{X})}{d\mathcal{X}}\right|_{\xi=0} = n(x,y)}$ for a uniform density elliptical beam with radii r_x, r_y and density $\lambda/(q\pi r_x r_y)$

Apply these results to calculate ϕ interior to a uniform density elliptical beam.

$$\phi = -\frac{q r_x r_y}{4\varepsilon_0}\int_0^\infty d\xi \frac{\eta(\mathcal{X})}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

$$\mathcal{X} = \frac{x^2}{r_x^2+\xi}+\frac{y^2}{r_y^2+\xi}$$ if $\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}<1$, then

$$\mathcal{X} = \frac{x^2}{r_x^2+\xi}+\frac{y^2}{r_y^2+\xi}<1 \text{ for all } 0\le\xi<\infty$$

Using this and the result above for $\eta(\mathcal{X})$, ϕ inside the elliptical beam is:

$$\phi = -\frac{q r_x r_y}{4\varepsilon_0}\int_0^\infty d\xi\,\frac{\lambda}{q\pi r_x r_y}\left[\frac{x^2}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}}+\frac{y^2}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}}\right]$$

$$\phi = \frac{-\lambda}{4\pi\epsilon_0}\left\{ x^2\int_0^\infty d\xi \frac{1}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}} + y^2\int_0^\infty d\xi \frac{1}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}} \right\}$$

Using Mathematica or Integral tables:

$$\int_0^\infty d\xi \frac{1}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}} = \frac{2}{r_x(r_x+r_y)}$$

$$\int_0^\infty d\xi \frac{1}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}} = \frac{2}{r_y(r_x+r_y)}$$

Hence

$$\phi = \frac{-\lambda}{2\pi\epsilon_0}\left\{ \frac{x^2}{r_x(r_x+r_y)} + \frac{y^2}{r_y(r_x+r_y)} \right\} + \text{const}$$

since an overall constant can always be added to ϕ (The integral has a reference choice $\phi(x=y=0)=0$ built in.).

The steps introduced in this proof can also be used to show that:

$$\left\langle x\frac{\partial\phi}{\partial x}\right\rangle_\perp = \frac{-\lambda}{4\pi\epsilon_0}\frac{r_x}{r_x+r_y}$$

$$\left\langle y\frac{\partial\phi}{\partial y}\right\rangle_\perp = \frac{-\lambda}{4\pi\epsilon_0}\frac{r_y}{r_x+r_y}$$

$\lambda = q\int d^2x\, n$

$r_x \equiv 2\langle x^2\rangle^{1/2}$

$r_y \equiv 2\langle y^2\rangle^{1/2}$

for any elliptic symmetry density profile $n(x,y) = n(x^2/r_x^2 + y^2/r_y^2)$. In the intro. lectures these results were employed to show that the KV envelope equations with evolving emittances can be applied to elliptic symmetry beams. This result was first demonstrated by Sacherer: [IEEE Trans Nucl. Sci. 18, 1105 (1971)]

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Appendix B

Canonical Transformation of the KV Distribution

The single-particle equations of motion can be derived from the Hamiltonian: $\left(\frac{d\vec{x}_\perp}{ds} = \frac{\partial H}{\partial \vec{x}_\perp'}, \ \frac{d\vec{x}_\perp'}{ds} = -\frac{\partial H}{\partial \vec{x}_\perp}\right)$

$$\boxed{\begin{aligned} H_\perp(x,y,x',y',s) = \frac{1}{2}x'^2 + \left[\kappa_x(s) - \frac{2Q}{r_x(s)[r_x(s)+r_y(s)]}\right]\frac{x^2}{2} \\ + \frac{1}{2}y'^2 + \left[\kappa_y(s) - \frac{2Q}{r_y(s)[r_x(s)+r_y(s)]}\right]\frac{y^2}{2} \end{aligned}}$$

Perform a canonical transform to new variables $\mathcal{X}, \mathcal{Y}, \mathcal{X}', \mathcal{Y}'$ using the generating function

$$F_2(x,y,\mathcal{X}',\mathcal{Y}') = \frac{x}{w_x}\left[\mathcal{X}' + \frac{x w_x'}{2}\right] + \frac{y}{w_y}\left[\mathcal{Y}' + \frac{y w_y'}{2}\right]$$

Then:

$$\boxed{\begin{aligned} \mathcal{X} &= \frac{\partial F_2}{\partial \mathcal{X}'} = \frac{x}{w_x} \\ \mathcal{Y} &= \frac{\partial F_2}{\partial \mathcal{Y}'} = \frac{y}{w_y} \end{aligned}}$$

$$x' = \frac{\partial F_2}{\partial x} = \frac{1}{w_x}\left(\mathcal{X}' + x w_x'\right)$$

$$y' = \frac{\partial F_2}{\partial y} = \frac{1}{w_y}\left(\mathcal{Y}' + y w_y'\right)$$

Ref:
R.C. Davidson, "Physics of Nonneutral Plasmas" Addison-Wesley, 1990

Comment!
Here, $\mathcal{X}' \neq \frac{d}{ds}\mathcal{X}$, $\mathcal{X}'$ merely denotes the conjugate variable to $\mathcal{X}$
Also, $\mathcal{X}, \mathcal{X}'$ both have dim: meters$^{1/2}$

and solving for $\mathcal{X}', \mathcal{Y}'$:

$$\boxed{\begin{aligned} \mathcal{X}' &= w_x x' - x w_x' \\ \mathcal{Y}' &= w_y y' - y w_y' \end{aligned}}$$

The Courant-Snyder invariants are then simply expressed:

$$\boxed{\begin{aligned} \mathcal{X}^2 + \mathcal{X}'^2 &= \text{const} \\ \mathcal{Y}^2 + \mathcal{Y}'^2 &= \text{const} \end{aligned}}$$

One can show from the transformations that:

$$\boxed{\begin{aligned} dx\,dy &= w_x w_y \, dX\,dY \\ dx'dy' &= \frac{dX'dY'}{w_x w_y} \\ dx\,dy\,dx'dy' &= dX\,dY\,dX'dY' \; {}^{*} \end{aligned}}$$

* Property of canonical transforms in general. — Results from structure of Generating Function

Therefore, the distribution in transformed phase space variables is the same as for the original variables:

$$\boxed{\begin{aligned} f_\perp(X,Y,X',Y',s) &= f_\perp(x,y,x',y',s) \\ &= \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y}\,\delta\left[\frac{X^2+X'^2}{\varepsilon_x} + \frac{Y^2+Y'^2}{\varepsilon_y} - 1\right] \end{aligned}}$$

Now examine the density:

$$n(x,y) = \int dx'dy' \, f_\perp = \int \frac{dX'dY'}{w_x w_y} f_\perp$$

$$U_x = X'/\sqrt{\varepsilon_x}, \quad U_y = Y'/\sqrt{\varepsilon_y}$$

$$dU_x dU_y = \frac{dX'dY'}{\sqrt{\varepsilon_x \varepsilon_y}}$$

$$r_x = \sqrt{\varepsilon_x}\, w_x, \quad r_y = \sqrt{\varepsilon_y}\, w_y$$

$$\boxed{n = \frac{\lambda}{q\pi^2 r_x r_y}\int dU_x dU_y \, \delta\left[U_x^2 + U_y^2 - \left(1 - \frac{X^2}{\varepsilon_x} - \frac{Y^2}{\varepsilon_y}\right)\right]}$$

Exploit the cylindrical symmetry:

$$U_\perp^2 = U_x^2 + U_y^2$$

$$dU_x\,dU_y = d\psi\, U_\perp dU_\perp = d\psi\,\frac{dU_\perp^2}{2}$$

$$n(x,y) = \frac{\lambda}{q\pi^2 r_x r_y}\int_0^{2\pi} d\psi \int_0^{\infty}\frac{dU_\perp^2}{2}\,\delta\left[U_\perp^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right]$$

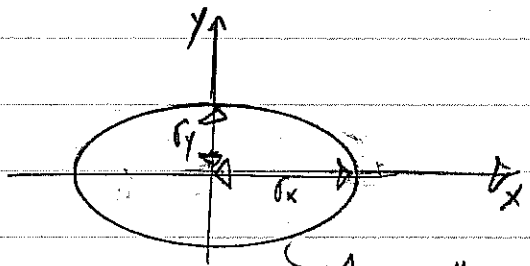
Thus:

$$\boxed{\begin{aligned} n(x,y) &= \frac{\lambda}{q\pi r_x r_y}\int_0^{\infty} dU_\perp^2\,\delta\left[U_\perp^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] \\ &= \begin{cases} \dfrac{\lambda}{q\pi r_x r_y} = \hat{n}\,; & \dfrac{x^2}{r_x^2} + \dfrac{y^2}{r_y^2} < 1 \\ 0\,; & \dfrac{x^2}{r_x^2} + \dfrac{y^2}{r_y^2} > 1 \end{cases} \end{aligned}}$$

Showing that the singular KV distribution yields the required uniform density beam of elliptical cross-section.

Note

$$\hat{n} = \frac{\lambda}{q\pi r_x r_y}$$



Area ellipse $= \pi r_x r_y$

$\lambda = q\hat{n}\,\pi r_x r_y$ for uniform density.

S. M. Lund 8/4/

An interesting footnote to this appendix is that an infinity of Generating functions can be used to transform the KV distribution in standard quadratic form:

$$f_\perp \sim \delta[\underline{X}'^2 + \underline{Y}'^2 + \underline{X}^2 + \underline{Y}^2 - \text{const}]$$

to other sets of variables. This will generate other distributions with KV form for skew coupling and other effects. It would not be logical to label such distributions as "new" as has been done in the literature. However, identifying physically relevant transforms has practical value.

Inverse transform:

$$x = w_x \bar{X}$$

$$w_x x' = \bar{X}' + x w_x' \quad \Rightarrow \quad x' = \frac{\bar{X}' + w_x' \bar{X}}{w_x}$$

$$\boxed{\begin{array}{ll} x = w_x \bar{X} & x' = \bar{X}'/w_x + w_x' \bar{X} \\ y = w_y \bar{Y} & y' = \bar{Y}'/w_y + w_y' \bar{Y} \end{array}}$$

Next,

$$\frac{d}{ds} \bar{X} = \frac{x'}{w_x} - \frac{x\, w_x'}{w_x^2}$$

$$= \frac{\bar{X}'}{w_x^2} + \cancel{\frac{w_x' \bar{X}}{w_x}} - \cancel{\frac{w_x' \bar{X}}{w_x}} = \frac{\bar{X}'}{w_x^2}$$

Thus,

$$\boxed{\begin{array}{l} \dfrac{d}{ds} \bar{X} = \dfrac{\bar{X}'}{w_x^2} \\ \dfrac{d}{ds} \bar{Y} = \dfrac{\bar{Y}'}{w_y^2} \end{array}}$$

$$\frac{d}{ds} \bar{X}' = \cancel{w_x' x'} + w_x x'' - \cancel{x' w_x'} + x\, w_x''$$

$$\Rightarrow \boxed{\begin{array}{l} x'' = \dfrac{\frac{d}{ds} \bar{X}'}{w_x} + w_x'' \bar{X} \\ y'' = \dfrac{\frac{d}{ds} \bar{Y}'}{w_y} + w_y'' \bar{Y} \end{array}}$$

Apply in Eqn of motion:

$$x'' + \kappa_x x - \frac{2Q x}{(r_x + r_y) r_x} = 0$$

$$\frac{\frac{d}{ds}\bar{X}'}{W_x} + W_x''\bar{X} + \kappa_x W_x \bar{X} - \frac{2Q W_x \bar{X}}{(r_x+r_y)r_x} = 0$$

$$\frac{d}{ds}\bar{X}' + W_x\underbrace{\left(W_x'' + \kappa_x W_x - \frac{2Q W_x}{(r_x+r_y)r_x}\right)}_{= \frac{1}{W_x^3} \text{ from } W_x \text{ eqn}}\bar{X} = 0$$

$$\boxed{\begin{aligned} &\frac{d}{ds}\bar{X}' + \frac{1}{W_x^2}\bar{X} = 0, \qquad \frac{d}{ds}\bar{X} = \frac{\bar{X}'}{W_x^2} \\ &\frac{d}{ds}\bar{Y}' + \frac{1}{W_y^2}\bar{Y} = 0, \qquad \frac{d}{ds}\bar{Y} = \frac{\bar{Y}'}{W_y^2} \end{aligned}}$$

Following Davidson, these eqns can be solved

$$\bar{X}(s) = \bar{X}_i \cos\psi_x(s) + \bar{X}_i' \sin\psi_x(s)$$

$$\psi_x(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{W_x^2(\tilde{s})}$$

etc. This also demonstrates explicitly the C.S. invariant

$$\bar{X}^2 + \bar{X}'^2 = \text{const.}$$