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June 2006

Elliptical Beam Envelope Equations -- cont'd

Current limits

A. Axisymmetric
   1. Solenoids
   2. Einzel lens
B. Quadrupolar
   1. Derivation of envelope equations with elliptic symmetry
   2. Current limit using fourier transform method
   3. Alternative methods

Centroid motion

YESTERDAY, WE DERIVED THE PARAXIAL RAY EQUATION FOR PARTICLES IN AXISYMMETRIC SYSTEMS:

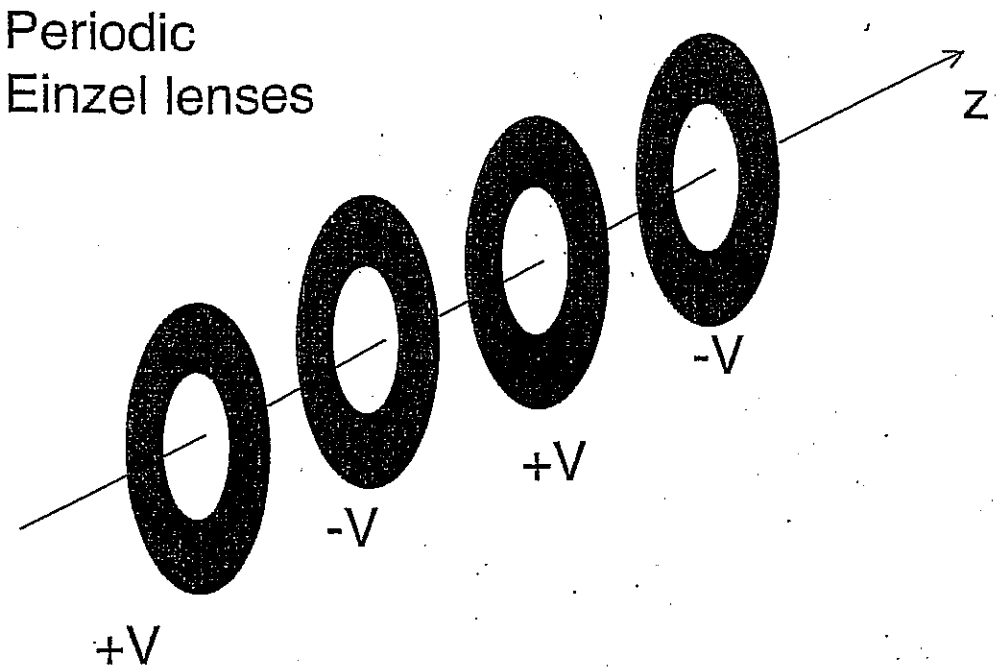
$$\underbrace{r'' + \frac{\gamma'}{\beta^2\gamma} r'}_{\text{INERTIAL}} + \underbrace{\frac{\gamma''}{2\beta^2\gamma}}_{E_r} + \underbrace{\left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r}_{\substack{V_\theta B_z \\ -\text{CENTRIFUGAL}}} - \underbrace{\left(\frac{p_\theta}{\gamma\beta m c}\right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}}_{\text{SELF-FIELD}} = 0$$

$$\theta' = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{\omega_c}{2\gamma\beta c}$$ ← CONSTANCY & DEFINITION OF CANONICAL MOMENTUM

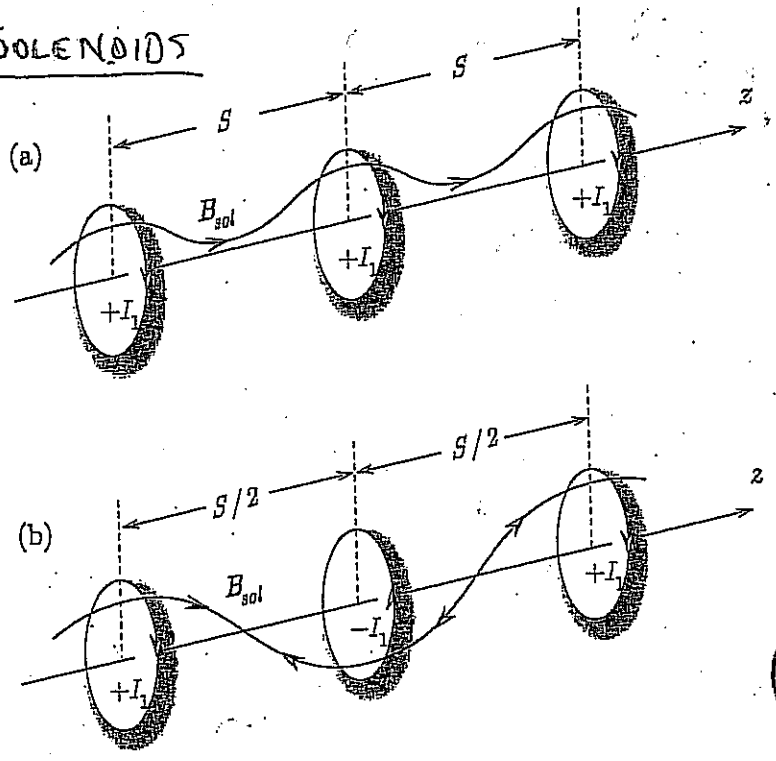
ENVELOPE EQUATION FOR AXISYMMETRIC BEAM

$$r_b'' + \frac{\gamma' r_b'}{\beta^2\gamma} + \frac{\gamma''}{2\beta^2\gamma} r_b + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b - \frac{4\langle p_\theta\rangle^2}{(\gamma m\beta c)^2 r_b^3} - \frac{\epsilon_r^2}{r_b^3} - \frac{Q}{r_b} = 0$$

$$\epsilon_r^2 \equiv 4\left(\langle r^2\rangle\langle r'^2\rangle - \langle r r'\rangle^2 + \langle r^2\rangle\langle r^2\theta'^2\rangle - \langle r^2\theta'\rangle^2\right)$$



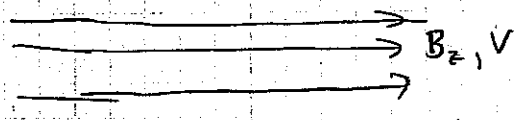
PERIODIC SOLENOIDS



(FIGURE FROM DAVIDSON & QIN, 2003) p. 55 "PHYSICS OF INTENSE (HIGH-CURRENT) PARTICLE BEAMS IN HIGH ENERGY ACCELERATORS"

Figure 3.2. Schematic of magnet sets producing a periodic focusing solenoidal field with axial periodicity length S. In Fig. 3.2 (a), successive coils are spaced by S and have the same current polarity $+I_1, +I_1, \cdots$. In Fig. 3.2 (b), successive coils are spaced by $S/2$ and have alternating current polarities $+I_1, -I_1, +I_1, \cdots$.

SOLENOIDAL FOCUSING



Let $\gamma' = \gamma'' = 0$

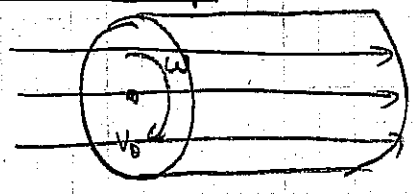
FOR MAXIMUM TRANSPORT $P_\theta = 0$ & $\varepsilon_r^2 = 0$

$$\Rightarrow \quad r_b'' + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b = \frac{Q}{r_b}$$

FOR A MATCHED BEAM:

$$Q_{max} = \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r_b^2$$

HEURISTICALLY:



$V_\theta = \omega r$

$$m\omega^2 r + Q m V_z^2 \left(\frac{r}{r_b^2}\right) = q \underbrace{\omega r}_{V_\theta} B_z$$

centrifugal force — SPACE CHARGE FORCE — MAGNETIC FORCE INWARD

$$\Rightarrow \quad \omega^2 + \frac{QV^2}{r_b^2} = \omega\omega_c$$

$\omega\omega_c - \omega^2 =$ MAXIMUM WHEN $\omega = \frac{\omega_c}{2}$

$$\Rightarrow \quad Q_{max} = \left(\frac{\omega_c^2}{4}\right)\left(\frac{r_b^2}{V^2}\right)$$

EINZEL LENS

J. BARNARD

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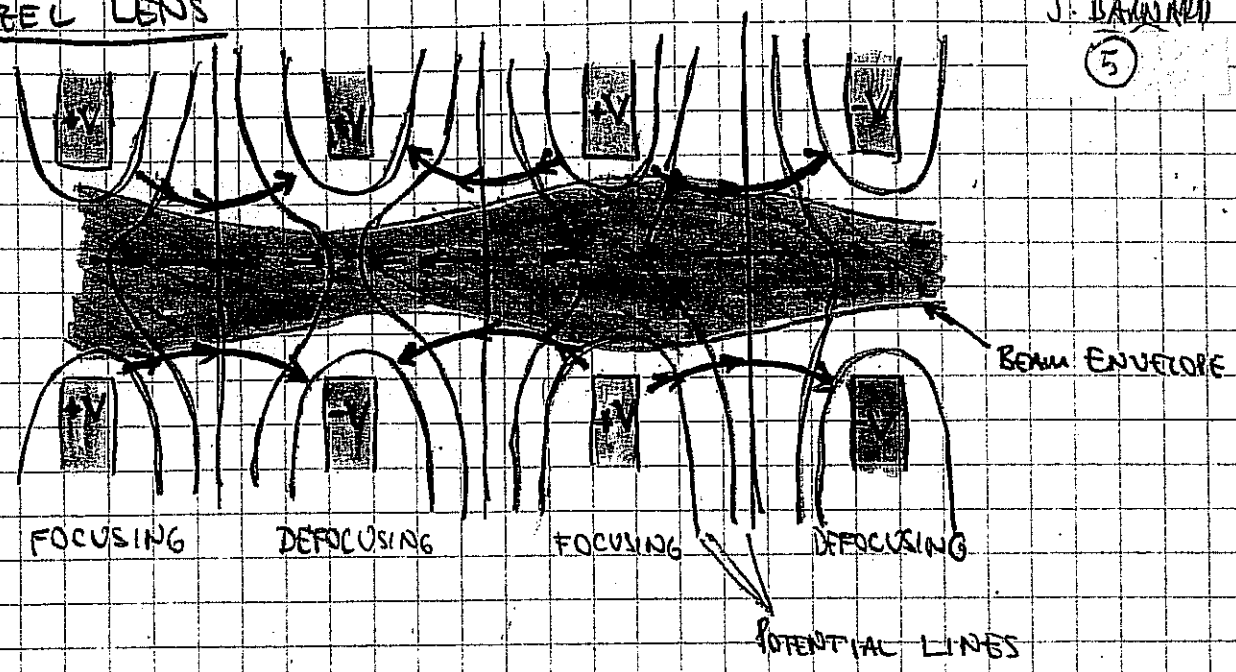

FOCUSING OCCURS AT LARGE RADIUS THAN DEFOCUSING

$\Rightarrow$ NET INWARD FORCE

EINZEL LENS – ANALYSIS (DERIVATION FROM ED LEE)

NOW, LET $\omega_c = \langle P_\theta \rangle = \varepsilon_r^2 = 0$

$$\Rightarrow r_b'' + \frac{\gamma'}{\beta^2\gamma} r_b' + \frac{\gamma''}{2\beta^2\gamma} r_b - \frac{Q}{r_b} = 0$$

ALSO ASSUME $\beta \ll 1$, NON-RELATIVISTIC BEAM $\rightarrow \gamma' \simeq \beta\beta'$, $\rightarrow \gamma'' \simeq \beta'^2 + \beta''\beta$

$$r_b'' + \frac{\beta'}{\beta} r_b' + \left[\frac{1}{2}\frac{\beta'^2}{\beta^2} + \frac{1}{2}\frac{\beta''}{\beta}\right] r_b - \frac{Q}{r_b} = 0$$

To eliminate r_b' term try substitution

$$r_b = \left(\frac{\beta_0}{\beta}\right)^{1/2} R$$

$$r_b' = \left(\frac{\beta_0}{\beta}\right)^{1/2} R' - \frac{1}{2}\left(\frac{\beta}{\beta_0}\right)^{-3/2} \frac{R}{\beta_0}\beta'$$

$$r_b'' = \left(\frac{\beta_0}{\beta}\right)^{1/2} R'' - \left(\frac{\beta}{\beta_0}\right)^{-\frac{3}{2}} \frac{R'}{\beta_0}\beta' + \frac{3}{4}\left(\frac{\beta}{\beta_0}\right)^{-\frac{5}{2}} \frac{R}{\beta_0^2}\beta'^2 - \frac{1}{2}\left(\frac{\beta}{\beta_0}\right)^{-\frac{3}{2}} \frac{R}{\beta_0}\beta''$$

$$\Rightarrow \left(\frac{\beta_0}{\beta}\right)^{1/2} R'' + \frac{3}{4}\left(\frac{\beta}{\beta_0}\right)^{-\frac{5}{2}} \frac{\beta'^2}{\beta_0^2} R = \frac{Q}{R}\left(\frac{\beta}{\beta_0}\right)^{1/2}$$

$$\Rightarrow \boxed{R'' = \frac{Q}{R}\left(\frac{\beta}{\beta_0}\right) - \frac{3}{4}\left(\frac{\beta'}{\beta}\right)^2 R}$$

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EINZEL LENS - CONTINUED

MODEL: LET $\phi = \phi_0 \cos\left(\frac{\pi z}{L}\right)$

$$\frac{1}{2} m v^2 + q\phi = \text{constant}$$

$$\Rightarrow \quad v^2 = v_0^2 + \frac{2q\phi}{m} \cos\left(\frac{\pi z}{L}\right)$$

$$v' = \frac{-q\phi_0}{mv}\left(\frac{\pi}{L}\right) \sin\left(\frac{\pi z}{L}\right)$$

If $\left(\frac{2q\phi_0}{m}\right) << v_0^2$: $\left(\frac{\beta'}{\beta}\right)^2 \cong \left(\frac{q\phi_0}{m v_0^2}\right)^2 \left(\frac{\pi}{L}\right)^2 \sin^2\left(\frac{\pi z}{L}\right)$

FOR EQUILIBRIUM LOOK AT D.C. COMPONENT: $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$R'' = 0 \quad \Rightarrow \quad \frac{Q}{\bar{R}} = \frac{3}{4} \overline{\left(\frac{\beta'}{\beta}\right)^2}\, \bar{R}$$

$$R \equiv \left(\frac{\beta}{\beta_0}\right)^{1/2} r_b \quad \Rightarrow \quad \bar{R} = r_b$$

$$\overline{\left(\frac{\beta'}{\beta}\right)^2} = \frac{1}{2}\left(\frac{q\phi_0}{m v_0^2}\right)^2 \left(\frac{\pi}{L}\right)^2$$

$$\Rightarrow \quad Q_{max} = \frac{3\pi^2}{8}\left(\frac{q\phi_0}{m v_0^2}\right)^2 \left(\frac{r_b}{L}\right)^2$$

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QUADRUPOLE ANALYSIS

J BRANNON

EQUATION OF MOTION

RETURN TO x, y COORDINATES

$$x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)x' = \frac{-q}{\gamma^3 m v_z^2}\frac{\partial \varphi}{\partial x} \pm \begin{cases} \frac{qB'}{\gamma m v_z^2}x & \text{for magnetic quads} \\ \frac{qE'}{\gamma m v_z^2}x & \text{for electric quads} \end{cases}$$

Let $\frac{\gamma m v_z}{q} = \frac{p}{q} \equiv [B\rho] \equiv$ RIGIDITY

$$y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)y' = \frac{-q}{\gamma^3 m v_z^2}\frac{\partial \varphi}{\partial y} \mp \begin{cases} \frac{B'}{[B\rho]}y & \text{magnetic} \\ \frac{qE'}{\gamma m v_z^2}y & \text{electric} \end{cases}$$

ENVELOPE EQUATION

$$r_x^2 = 4\langle x^2\rangle; \quad r_y^2 = 4\langle y^2\rangle$$

$$r_x' = \frac{4\langle x x'\rangle}{r_x}$$

$$r_x'' = \frac{4\langle x x''\rangle}{r_x} + \frac{\varepsilon_x^2}{r_x^3}; \qquad \varepsilon_x^2 = 16\left(\langle x^2\rangle\langle x'^2\rangle - \langle x x'\rangle^2\right)$$

$$r_y'' = \frac{4\langle y y''\rangle}{r_y} + \frac{\varepsilon_y^2}{r_y^3} \qquad \varepsilon_y^2 = 16\left(\langle y^2\rangle\langle y'^2\rangle - \langle y y'\rangle^2\right)$$

For magnetic focusing:

$$r_x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\, r_x' + \frac{4q}{\gamma^3 m v_z^2}\frac{\langle x \frac{\partial \varphi}{\partial x}\rangle}{r_x} \pm \frac{B'}{[B\rho]}r_x - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z)\, r_y' + \frac{4q}{\gamma^3 m v_z^2}\frac{\langle y \frac{\partial \varphi}{\partial y}\rangle}{r_y} \pm \frac{B'}{[B\rho]}r_x - \frac{\varepsilon_y^2}{r_y^3} = 0$$

(for electric focusing $\frac{B'}{[B\rho]} \to \frac{qE'}{\gamma m v_z^2}$)

SPACE CHARGE TERM WITH ELLIPTICAL SYMMETRY

NOW DEFOCUSING IN ONE DIRECTION AND FOCUSING IN THE OTHER $\Rightarrow$ RADIAL SYMMETRY SHOULD BE REPLACED BY ELLIPTICAL SYMMETRY:

$$\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$$

CAN BE SHOWN THAT

$$\left\langle x \frac{\partial \phi}{\partial x} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_x}{r_x + r_y}$$

$$\left\langle y \frac{\partial \phi}{\partial y} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_y}{r_x + r_y}$$

DEFINING $Q = \dfrac{2q\lambda}{4\pi\varepsilon_0 \gamma^3 m v_z^2}$

$$\boxed{\begin{aligned} r_x'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z) r_x' - \frac{2Q}{r_x + r_y} \mp \frac{B'}{[B\rho]} r_x - \frac{\varepsilon_x^2}{r_x^3} &= 0 \\ r_y'' + \frac{1}{\gamma v_z}\frac{d}{ds}(\gamma v_z) r_y' - \frac{2Q}{r_x + r_y} \pm \frac{B'}{[B\rho]} r_y - \frac{\varepsilon_y^2}{r_y^3} &= 0 \end{aligned}}$$

(for Electric Focusing $\dfrac{B'}{[B\rho]} \to \dfrac{qE'}{\gamma m v_z^2}$)

SPACE CHARGE TERM WITH ELLIPTICAL SYMMETRY II

ELLIPTICAL SYMMETRY: $\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$

CAN BE SHOWN THAT $\left\langle x \frac{\partial \varphi}{\partial x} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_x}{r_x + r_y}$

(Sacherer, 1971)

& $\left\langle y \frac{\partial \varphi}{\partial y} \right\rangle = \frac{-\lambda}{4\pi\varepsilon_0} \frac{r_y}{r_x + r_y}$

OUTLINE OF PROOF: (from R. Ryne)

Let $\chi = \frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s}$

DEFINE $\eta(\chi)$ such that $\rho(x,y) = \left.\frac{d\eta(\chi)}{d\chi}\right|_{s=0} = \hat{\rho}(\chi)\big|_{s=0}$

So $\rho = \hat{\rho}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right) = \hat{\rho}(\chi)\big|_{s=0}$

DEFINE $\Psi(x,y) = \frac{-r_x r_y}{4\varepsilon_0} \int_0^\infty \frac{\eta(\chi)\, ds}{\sqrt{r_x^2+s}\,\sqrt{r_y^2+s}}$

It follows that $\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = -\frac{\rho}{\varepsilon_0}$ AND SO IS A SOLUTION OF POISSON'S EQUATION (since $\Psi \to 0$ as $x, y \to \infty$)

WHAT IS $\left\langle x \frac{\partial \Psi}{\partial x} \right\rangle$?

$$\left\langle x \frac{\partial \Psi}{\partial x} \right\rangle = \frac{-r_x r_y}{4\lambda\varepsilon_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, x\, \rho(x,y) \int_0^\infty \frac{\eta' \frac{\partial \chi}{\partial x}\, ds}{\sqrt{r_x^2+s}\,\sqrt{r_y^2+s}}$$

where $\lambda = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \rho(x,y)$

So $$\left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-2 r_x r_y}{4 \lambda \varepsilon_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; x^2 \hat{\rho}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right) \int_0^{\infty} \frac{\hat{\rho}\left(\frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s}\right) ds}{(r_x^2+s)^{3/2}(r_y^2+s)^{1/2}}$$

Let $r\cos\theta = \dfrac{x}{\sqrt{r_x^2+s}}$ $\qquad r\sin\theta = \dfrac{y}{\sqrt{r_y^2+s}}$

$\det J = \sqrt{r_x^2+s}\,\sqrt{r_y^2+s}\; r$ $\qquad$ where J is the Jacobian $dx\,dy = \det J \cdot dr\,d\theta$

$$\Rightarrow \left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-r_x r_y}{\lambda 2 \varepsilon_0} \int_0^{\infty} dr \int_0^{2\pi} d\theta \int_0^{\infty} ds \; r^3 \hat{\rho}(r^2)\, \rho\left(\frac{r_x^2+s}{r_x^2} r^2\cos^2\theta + \frac{r_y^2+s}{r_y^2} r^2 \sin^2\theta\right)$$

Let $$r'^2 = \frac{r_x^2+s}{r_x^2} r^2\cos^2\theta + \frac{r_y^2+s}{r_y^2} r^2 \sin^2\theta$$
$$= r^2\left[1 + s\left(\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}\right)\right]$$

with r fixed $\quad 2r'dr' = r^2\left(\dfrac{\cos^2\theta}{r_x^2} + \dfrac{\sin^2\theta}{r_y^2}\right) ds$

$$\Rightarrow \left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-r_x r_y}{2\lambda \varepsilon_0} \int_0^{\infty} dr \int_0^{2\pi} d\theta \int_r^{\infty} dr' \; \frac{2r'dr'\, r^3 \rho(r^2)\, \hat{\rho}(r'^2) \cos^2\theta}{r^2\left(\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}\right)}$$

$$\int_0^{2\pi} \frac{\cos^2\theta \; d\theta}{\frac{\cos^2\theta}{r_x^2} + \frac{\sin^2\theta}{r_y^2}} = \frac{2\pi\, r_x^2 r_y}{r_x + r_y}$$

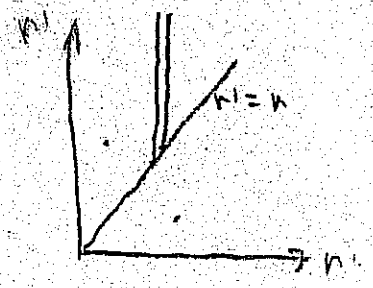
$$\Rightarrow \left\langle x \frac{\partial \psi}{\partial x} \right\rangle = \frac{-r_x^3 r_y^2}{\lambda 2\pi \varepsilon_0 (r_x + r_y)} \int_0^{\infty} dr\, 2\pi r\, \hat{\rho}(r^2) \int_r^{\infty} dr'\, 2\pi r' \rho(r'^2)$$

Recall $$\lambda = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\,(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\; \hat{\rho}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)$$

Let $\dfrac{x}{r_x} = r\cos\theta \quad \dfrac{y}{r_y} = r\sin\theta \quad \det J = r_x r_y r$

$$\Rightarrow \lambda = \int_0^{\infty} dr \int_0^{2\pi} d\theta\; \rho(r^2)\, r_x r_y r = 2\pi r_x r_y \int_0^{\infty} dr\, r\, \rho(r^2)$$

Now $\int_0^\infty r\,dr \int_r^\infty r'\,dr'\ \hat{\rho}(r^2)\,\hat{\rho}(r'^2) = \frac{1}{2}\underbrace{\int_0^\infty dr\ r\,\hat{\rho}(r^2)}_{\lambda/2\pi r_x r_y}\ \underbrace{\int_0^\infty dr'\ r'\,\hat{\rho}(r'^2)}_{\lambda/2\pi r_x r_y}$



$\hat{\rho}(r^2)\,\hat{\rho}(r'^2)$ is the same on switching r & r'.

So $$\left\langle x\frac{\partial\psi}{\partial x}\right\rangle = \frac{-r_x^3 r_y}{\lambda\, 2\pi\epsilon_0 (r_x - r_y)}\left[\frac{1}{2}\,\frac{\lambda}{2\pi r_x r_y}\cdot\frac{\lambda}{2\pi r_x r_y}\right]$$

$$= \frac{-\lambda}{4\pi\epsilon_0}\,\frac{r_x}{r_x + r_y}$$

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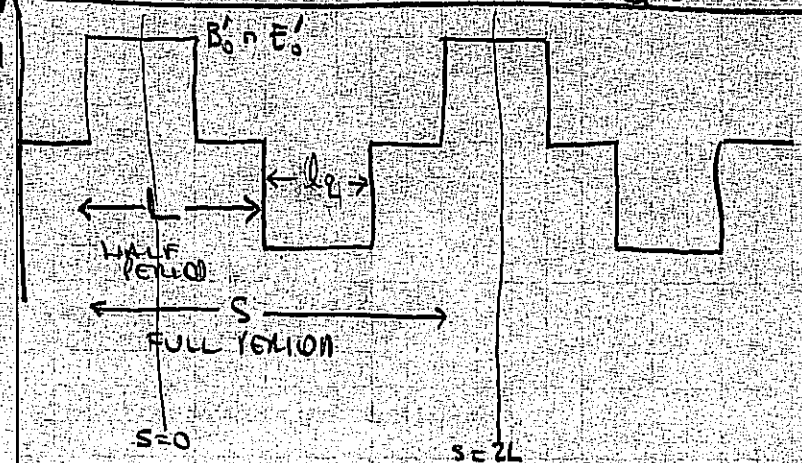
CURRENT LIMIT FOR QUADRUPOLES (FODO)



$$k = \begin{cases} \dfrac{B_0'}{[B\rho]} & \text{MAGNETIC} \\ \dfrac{qE_0'}{\gamma m v_z^2} & \text{ELECTRIC} \end{cases}$$

$$r_x'' + k f(s) r_x - \frac{2Q}{r_x + r_y} = 0$$

(NOTE WE HAVE SET $\epsilon = 0$).

$$r_y'' - k f(s) r_y - \frac{2Q}{r_x + r_y} = 0$$

$$f(s) = \begin{cases} 1 & 0 < s < \eta L/2 \\ -1 & L - \frac{\eta L}{2} < s < L + \eta L/2 \\ 1 & 2L - \eta L/2 < s < 2L \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{1}{2L}\int_0^{2L} f(s) \cos\left(\frac{\pi s}{L}\right) ds = \left(\frac{2}{\pi}\right) \sin\left(\frac{\pi \eta}{2}\right)$$ = fourier amplitude at fundamental lattice frequency

Let $r_x = r_b\left(1 + \delta \cos\left(\frac{\pi s}{L}\right)\right)$

$r_y = r_b\left(1 - \delta \cos\left(\frac{\pi s}{L}\right)\right)$

Let $k f(s) \rightarrow k\left(\frac{2}{\pi}\right)\sin\left(\frac{\eta\pi}{2}\right) \cos\left(\frac{\pi s}{L}\right)$

COLLECTING "FAST" TERMS AND "SLOW" TERMS

$$\left[\frac{-\pi^2}{L^2} r_b \delta + k r_b \left(\frac{4 \sin\left(\frac{\eta\pi}{2}\right)}{\pi}\right)\right] \cos\left(\frac{\pi s}{L}\right) = 0 \quad \text{(fast)}$$

$$\delta k r_b \left(\frac{2 \sin\left(\frac{\eta\pi}{2}\right)}{\pi}\right) = \frac{Q}{r_b} \quad \text{(slow)}$$

Fast $\Rightarrow$ $\delta = \dfrac{2kL^2}{\pi^3} \sin\left(\dfrac{\eta\pi}{2}\right)$ & $Q_{max} \cong \dfrac{\eta^2 k^2 L^2}{\pi^2}\left(\dfrac{\sin\left(\frac{\eta\pi}{2}\right)}{\left(\frac{\eta\pi}{2}\right)}\right)^2 r_b^2$

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CONTINUOUS FOCUSING

$$r_x'' = -k_{\beta 0}^2 x + \frac{2Q}{r_x + r_y} - \frac{\epsilon^2}{r_x^2}$$

$$r_y'' = -k_{\beta 0}^2 y + \frac{2Q}{r_x + r_y} - \frac{\epsilon^2}{r_y^2}$$

CURRENT LIMIT BALANCES PERVEANCE & EXTERNAL FOCUSING ($r_x = r_y = r_b$):

$$k_{\beta 0}^2 r_b = \frac{Q_{max}}{r_b}$$

Effective $k_{\beta 0}^2$ FOR QUADRUPOLES FOUND FROM DOMINANT FOURIER COMPONENT

$$k_{\beta 0}^2 = \frac{\eta^2 k^2 L^2}{\pi^2} \left(\frac{\sin\left(\frac{\eta\pi}{2}\right)}{\frac{\eta\pi}{2}} \right)^2 \qquad \text{where } k = \frac{B'}{[B\rho]}$$

FOR CONTINUOUS FOCUSING: $k_{\beta 0}^2 = \frac{\sigma_0^2}{4L^2}$

ELIMINATING L:

$$Q_{max} = \frac{\eta k \sigma_0}{2\pi} \left(\frac{\sin\frac{\eta\pi}{2}}{\frac{\eta\pi}{2}} \right) r_b^2 \quad \leftarrow$$

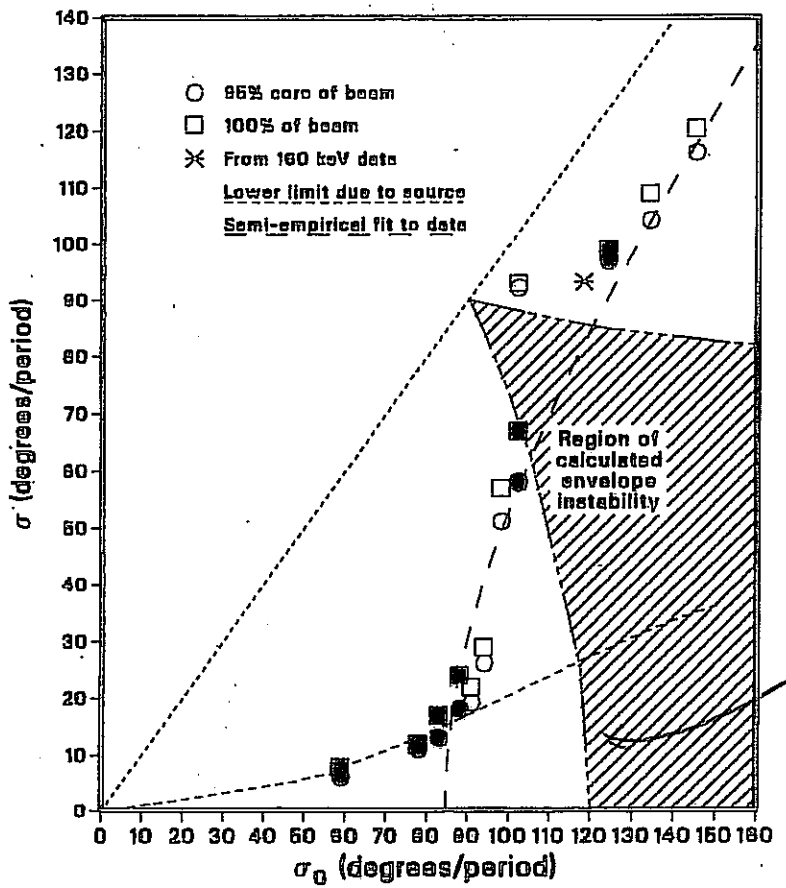
PERVEANCE LIMIT FOR FODO QUADRUPOLES

Envelope instabilities set upper limit on "single particle" phase advance σ_0

Experimental data (Tiefenback, 1986) from LBL's Single Beam Transport Experiment (SBTE)

$\sigma_0 < 85^0$

Experimental limits on beam stability in terms of σ and σ_0



SEE LUND & CHAWLA 2006, NIMPR-A, FOR HIGHER ORDER PARTICLE-LATTICE RESONANCES WHICH CLARIFIES $\sigma_0 = 85^0$ LIMIT

SEE STAUCKMEIR & REISER, PARTICLE ACCELERATORS 14, 227, (1984) & LUND & BUKH, PRSTAB, 7 024801 (2004)

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QUADRUPOLE CURRENT LIMIT — CONTINUED

$$Q_{max} \cong \frac{\eta k \Theta_0}{2\pi} \left(\frac{\sin \frac{\eta \pi}{2}}{\frac{\eta \pi}{2}} \right) r_b^2$$

$$\text{here } k = \begin{cases} \dfrac{dB/dx}{[B\rho]} \sim \dfrac{B}{[B\rho]\, r_p} & \text{(MAGNETIC QUAD FODO)} \\[2ex] \dfrac{q\, dE/dx}{\gamma m v_z^2} \sim \dfrac{2 q V_q}{\gamma m v_z^2 r_p^2} & \text{where } V_q = \frac{1}{2}\frac{dE}{dx} r_p^2 \quad \text{(ELECTRIC QUAD FODO)} \end{cases}$$

So

$$Q_{max} \cong \frac{\eta \Theta_0}{2\pi} \left(\frac{\sin \frac{\eta \pi}{2}}{\frac{\eta \pi}{2}} \right) \begin{cases} \dfrac{B\, r_b}{[B\rho]} \left[\dfrac{r_b}{r_p} \right] \\[2ex] \dfrac{2 q V_q}{\gamma m v_z^2} \left[\dfrac{r_b^2}{r_p^2} \right] \end{cases}$$

Summary of Current Limits From Different Focusing Methods

Einzel Lens

$$Q_{max} \cong \frac{3\pi^2}{8}\left(\frac{q\phi_0}{m v_0^2}\right)^2\left(\frac{r_b}{L}\right)^2$$

Solenoids

$$Q_{max} = \left(\frac{\omega_c r_b}{2\gamma\beta c}\right)^2$$

Quadrupole Focusing

$$Q_{max} \cong \frac{\eta\phi_0}{2\pi}\left(\frac{\sin\frac{\eta\pi}{2}}{\frac{\eta\pi}{2}}\right)\left[\frac{B r_b}{[B\rho]}\left[\frac{r_b}{r_p}\right] \quad \frac{2qV_q}{\gamma m v_z^2}\left[\frac{r_b^2}{r_p^2}\right]\right]$$

MAGNETIC

Electric

For Non-Relativistic Beams

$$\lambda_{max} \propto \frac{\phi_0^2}{V} \qquad \lambda_{max} \propto \frac{q}{m} B^2 r_p^2 \qquad \lambda_{max} \propto \begin{cases} B\, V^{1/2}\, r_p \\ V_q \end{cases}$$

Note
ϕ_0 = Voltage between Einzel lenses
V_q = Voltage on a quad relative to ground
V = particle energy / q

FIRST ORDER MOMENTS DESCRIBE "CENTROID" EVOLUTION

$$x'' = -\frac{\theta_0^2}{4L^2}x + \frac{Q(x-\langle x\rangle)}{r_b^2}$$

$$\frac{d}{ds}\langle x\rangle = \langle x'\rangle$$

$$\frac{d}{ds}\langle x'\rangle = \langle x''\rangle = -\frac{\theta_0^2}{4L^2}\langle x\rangle + \cancelto{0}{\frac{Q}{r_b^2}\left(\langle x\rangle - \langle x\rangle\right)}$$

$$\Rightarrow \frac{d^2\langle x\rangle}{ds^2} = -\frac{\theta_0^2}{4L^2}\langle x\rangle$$

$\Rightarrow$ MOTION OF CENTROID IS UNAFFECTED BY SPACE-CHARGE EXCEPT THROUGH IMAGE CHARGES

IMAGE FORCES (SUMMARY ONLY)

cf. REISER SEC 4.4.4

To linear order:

$$\frac{d^2\langle x\rangle}{ds^2} \cong \left(\frac{\theta_0^2}{4L^2} - \frac{Q}{r_p^2}\right)\langle x\rangle$$

IMAGES ATTRACT BEAM TO $1/r$ REDUCING FOCUSING FIELD

ENVELOPES ARE ALSO EFFECTED BY IMAGES (cf eg. Lee, Close, & Smith, Proc. 1987 IEEE PAC, p. 1126)

To lowest order:

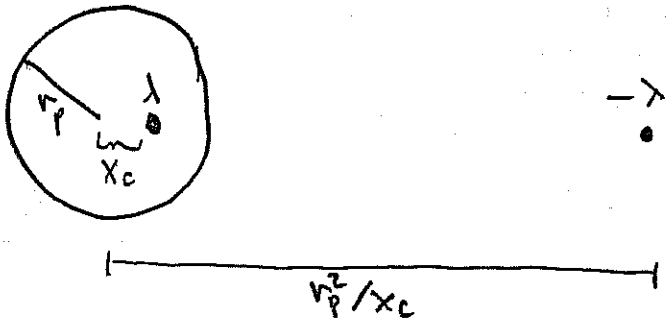
$$\frac{d^2 r_x}{ds^2} = \cdots + \frac{2Q}{r_x + r_y} + \frac{fQ}{r_p^2} r_x$$

$$\frac{d^2 r_y}{ds^2} = \cdots + \frac{2Q}{r_x + r_y} + \frac{fQ}{r_p^2} r_y$$

$$f = \frac{r_x^2 - r_y^2}{4r_p^2} + \frac{\langle x\rangle^2}{r_p^2}\left[1 + \frac{3}{2}\frac{(r_x^2 - r_y^2)}{r_p^2} + \frac{3}{8}\frac{(r_x^2 - r_y^2)^2}{r_p^4}\right]$$

IMAGE FORCES

Let $x_c = \langle x \rangle$



$$E_x = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{r_p^2/x_c - x_c} \simeq \frac{\lambda}{2\pi\varepsilon_0} \frac{x_c}{r_p^2}$$

IMAGE CURRENTS ALSO CAN GIVE RISE TO

$$|B_y| = \beta E_x$$

$$F_x = (1 - \beta^2)\, q E_x$$

$$\Rightarrow \frac{d^2\langle x \rangle}{ds^2} \cong \frac{-\sigma_0^2}{4L^2}\langle x \rangle + \frac{Q}{r_p^2}\langle x \rangle$$