Typically, the most serious numerical instability in PIC simulations of relativistic particle beams is the numerical Cherenkov instability, arising from coupling between electromagnetic and nonphysical beam modes. In recent papers we derived and solved electromagnetic dispersion relations for this instability in both finite difference time-domain (FDTD) and pseudo-spectral time-domain (PSTD) algorithms and successfully compared results with those of the Warp simulation code. Our PSTD analysis, focused on Haber's Pseudo-Spectral Analytical Time-Domain algorithm, provided several methods for suppressing the numerical Cherenkov instability. This was done by a combination of digital filtering at large wave-numbers and improved numerical balancing of transverse fields at smaller wave-numbers. Doing so is mechanically straightforward for PSTD algorithms, because currents and fields are known in Fourier space and readily can be rescaled by the desired k-dependent factors. In this talk we carry over such methods to FDTD algorithms without resorting to Fourier transforms. Digital filtering is achieved with the usual bilinear smoothing, and improved numerical balancing of transverse fields is accomplished using a similar process but with coefficients based on rational interpolation functions approximating the desired k-dependent factors. Results are very encouraging.


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Suppressing Numerical Cherenkov Instability in FDTD PIC Codes*

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Numerical Cherenkov Instability

- Serious issue in 2D, 3D EM PIC simulations of relativistic beams (accelerators, astrophysics, etc.)
  - Growth rates a large fraction of \( \left( \omega_p^2 k_\perp^2 \Delta t \right)^{1/3} \)
  - Mismatch between Lagrangian particle pusher, Eulerian field solver yields unstable beam-like normal modes
  - Typically addressed by heavy digital filtering

- New approach suppresses instability in FDTD PIC
  - Minor corrections to EM fields perpendicular to beam
  - Inexpensive, easy to implement
  - Works best with cubic interpolation, modest digital filtering
$k_x = 1$

\[ \frac{v \Delta t}{\Delta z} = 0.9, \ v \approx 1, \ \text{linear interpolation, no filtering} \]
# Field Interpolation Options

(B\textsubscript{y} – not shown – at cell centers)

<table>
<thead>
<tr>
<th>“Galerkin”</th>
<th>“Uniform”</th>
<th>“Momentum-Conserving”</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a.k.a. “Energy-Conserving”)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• interpolates from staggered grid</td>
<td>• interpolates from staggered grid</td>
<td>• first interpolate at grid nodes</td>
</tr>
<tr>
<td>• one order down in // direction</td>
<td>• same order in all directions</td>
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</tr>
</tbody>
</table>

**Field Interpolation Options**

- **“Galerkin”**
  - Interpolates from staggered grid
  - One order down in // direction

- **“Uniform”**
  - Interpolates from staggered grid
  - Same order in all directions

- **“Momentum-Conserving”**
  - First interpolate at grid nodes
  - Same order in all directions
Heavy Digital Filtering, Cubic Interpolation Reduce Growth

Acceptable stability only in narrow bands near “magic time steps”
FDTD High-\(\gamma\) Dispersion Relation

\[
C_0 + n \sum_m C_1 \csc \left[ (\omega - k'_z v) \frac{\Delta t}{2} \right] \\
+ n \sum_m C_2 \csc \left[ (\omega - k'_z v) \frac{\Delta t}{2} \right]^2 = 0
\]

with \(k'_z = k_z + m \frac{2\pi}{\Delta z}\) (aliases)

- Beam modes associated with \(C_1, C_2\) are numerical artifacts, trigger numerical Cherenkov instability
- \(C_2\) determined by \(E_\perp - vB_\perp\), which should vanish to lowest order at \(\omega \approx k_z v\) (but does not)
- Solution: Correct \(E_\perp, B_\perp\) in high-energy limit
Modify fields using operator (axial direction only)

- \( \Psi = \sum_i a_i (-\frac{1}{4}, \frac{1}{2}, -\frac{1}{4})^{i-1} \) with \( a_1 = 1 \)
- Choose Fourier Transform of \( \psi_E, \psi_B \) to set \( C_2 \sim 10^{-6} \)
- \( \{a_i\} \) of 4th order rational interpolation function for \( \psi_E / \psi_B \)
Ψ- Operators Yield Excellent Stability

Superior choices for \( \{a_i\} \) may produce even better results
Ψ-\(C_2\) Operator plus One-Pass Filter
Stabilize WARP LPA Simulations

Same simulations without \(\Psi-C_2\) Operators severely unstable, produce meaningless results
(Uniform interpolation simulation stable at “magic time step” \(\nu \Delta t / \Delta z = 0.5\))
\( \Psi-C_2 \) Operator plus One-Pass Filter Conserves Energy Well

Same simulations without \( \Psi-C_2 \) Operators do not conserve energy except near “magic time step”
References

• Software and other supporting material at http://hifweb.lbl.gov/public/BLAST/Godfrey/
  – RatIntCoef – Computes \( \{a_i\} \)
  – Resonances – Plots normal modes and resonances
  – FDTD – Solves numerical dispersion relation
  – Mathematica CDF Player needed (free)
  – Contact Brendan.Godfrey@ieee.org

• Additional details available at Journal of Computational Physics 267 (2014) 1–6
Anomalous Behavior at Galerkin

$v \Delta t / \Delta z = 0.5$ Due to Particle Count

- Simulations with 2, 6, 10, etc. particles/cell also very noisy
- Growth rates insensitive to particles/cell in x

(Six particles/cell in z; two in x)