PROBLEM SET #5

20 Points BARNARD & LUND, USPAS 2011

PROBLEM 1: WHEN LONGITUDINAL EMITTANCE IS INCLUDED IN THE NON-RELATIVISTIC LONGITUDINAL ENVELOPE EQUATION, DESCRIBING THE LENGTH $L$ OF A PULSE WITH PARABOLIC LINE CHARGE DENSITY UNDERGOING BUNCH COMPRESSION,

$$\frac{d^2L}{ds^2} = \frac{16\,\varepsilon_z^2}{L^3} + \frac{12\, g\, q\, Q_c}{4\pi\epsilon_0\, m v^2 L^2}$$

where $Q_c$ is the total charge in the bunch,

$\varepsilon_z^2 \equiv$ longitudinal emittance $= 25\left[\langle z^2\rangle\langle z'^2\rangle - \langle z z'\rangle^2\right]$.

SHOW THAT THE INITIAL VELOCITY TILT $\frac{\Delta V}{V_0}$ REQUIRED TO COMPRESS THE BEAM TO "STAGNATION" [i.e. to the point where $\frac{dL}{ds}=0$] IS GIVEN BY:

$$\frac{\Delta V^2}{V_0^2} = \frac{16\,\varepsilon_z^2}{L_0^2}\left[C^2 - 1\right] + \frac{24\, g\, q\, Q_c}{4\pi\epsilon_0\, L_0\, m V_0^2}\left[C - 1\right]$$

WHERE $L_0 = L$ at $s=0$,

$L_f = L$ at the stagnation point

& $C \equiv L_0/L_f$ = compression ratio $> 1$

& $\frac{\Delta V}{V_0} = -L_0' = \left.\frac{dL}{ds}\right|_{s=0}$ $V_0$ = longitudinal velocity of beam center

Problem 6, 30 Points

# TPR: Problem 1

S.M. Lund PV.

1/ Consider the driven harmonic oscillator equation for $U(\varphi)$:

$$\frac{d^2 U(\varphi)}{d\varphi^2} + \nu_0^2 U(\varphi) = \overbrace{A\cos(\nu\varphi) + B\sin(\nu\varphi)}^{\text{driving term.}}$$

$\nu$ = constant driving frequency.
$A, B$ constant amplitudes.

The general solution for $U(\varphi)$ can be expanded as

$$U(\varphi) = U_h(\varphi) + U_p(\varphi)$$

where $U_h$ is the general solution to the homogeneous equation:

$$\frac{d^2 U_h}{d\varphi^2} + \nu_0^2 U_h = 0$$

$$\Rightarrow U_h = C_1\cos(\nu_0\varphi) + C_2\sin(\nu_0\varphi)$$

$C_1, C_2$ constants

and $U_p$ is any particular solution to

$$\frac{d^2 U_p}{d\varphi^2} + \nu_0^2 U_p = A\cos(\nu\varphi) + B\sin(\nu\varphi)$$

a) For $\nu \neq \nu_0$ show that a solution $U_p$ exists proportional to the driving term and find the constant of proportionality.

b) Use the results of part a) to construct the solution ($\nu \neq \nu_0$) for $U(\varphi)$ satisfying the initial conditions at $\varphi = 0$:

$$U(\varphi=0) = U_0$$
$$\left.\frac{dU}{d\varphi}\right|_{\varphi=0} = \dot{U}_0 \qquad ; \qquad \frac{dU}{d\varphi} \equiv \dot{U}$$

c) Set $\nu = \nu_0 + \delta\nu$ and find the leading order form of the solution valid for $|\delta\nu|/\nu_0 \ll 1$ and $|\delta\nu\,\varphi| \ll 1$.
What does this limit imply on the amplitude of the particle oscillation as $\nu \to \nu_0$

d) What do these results imply for a general periodic forcing function:

$$\frac{d^2U(\varphi)}{d\varphi^2} + \nu_0^2\,U(\varphi) = f(\varphi) \quad \leftarrow \text{forcing function}$$
$$f(\varphi + 2\pi) = f(\varphi)$$

How does this fit in with the analysis of machine tunes carried out in the class notes?