PROBLEM SET 4

25 Points

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(1)

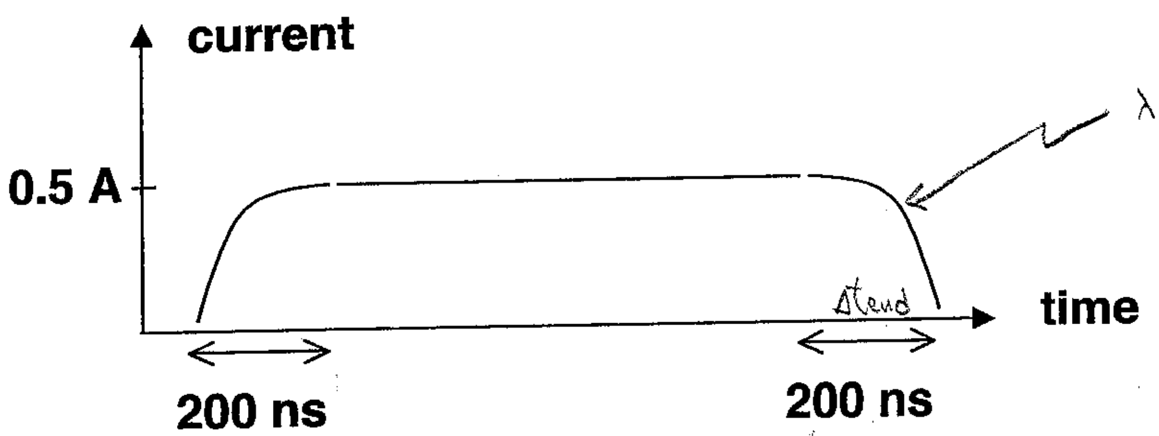
A 0.5 A, 2 MeV potassium+ (A=39) beam is injected into a transport section with a 25 cm half-lattice period L. There are 10 half-lattice periods in the transport section. The beam has a flattop of 1 μs. Assume $g = 2\ln\frac{r_p}{a} = 1.0$. ($r_p/a \cong 1.65$).

a). What is the beam velocity?

b). What is the space charge wave speed in the comoving beam frame?

c). What will the duration of the beam flattop be at the end of the transport section? (Assume no ear fields are applied, and assume a square pulse [with instantaneous rise and fall] for this calculation, at the beginning of the transport section.)

d). For a 200 ns long head and tail, with parabolic fall off (see figure), how large an "ear" field is required to keep the beam from spreading longitudinally?



$$\lambda = \left[1 - \frac{(z - z_{end})^2}{l_{end}^2}\right]\lambda_0$$

$$l_{end} = v_0 \Delta t_{end}$$

(2) 20 Points

20 POINTS

A velocity perturbation $\overline{z_1'}$ on a long coasting beam with center position $s = s_0$ has the initial form:

$$\overline{z_1'} = \delta \exp\left[\frac{-z^2}{\Delta^2}\right]$$

There is no initial density perturbation ($\lambda_1(z) = 0$). The space charge wave velocity of the beam is c_s and the beam velocity is v_0.

What is the density of the perturbation after the beam center propagates a distance s-s_0? What is the velocity perturbation $\overline{z_1'}$ for the same location of the beam center?

Sketch λ_1 and $\overline{z_1'}$ vs. z at a point when $s - s_0 > v_0\Delta/c_s$.

5 Points

(3) SHOW THAT $c_s^2 = \frac{gQ}{2} v_0^2$ FOR A NON-RELATIVISTIC BEAM.

(5 POINTS)

(HERE c_s^2 = THE SPACE-CHARGE WAVE SPEED)

Problem 4, 10 Points

TED Problem 4

S.M. Lund P4/

4/ For a continuous focusing channel with

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const.}$$

and a round, "matched" KV equilibrium beam with

$$\varepsilon_x = \varepsilon_y$$

$$r_x = r_y = r_b = \text{const}$$

a) Solve the KV envelope equation for the beam radius r_b in terms of the Perveance Q, $k_{\beta 0}$, and ε_x.

b) Solve for the zero space-charge amplitude function $w_0 = w_{0x} = w_{0y}$

c) Apply the **integral form of the phase advance formulas** for a matched beam:

σ_{0x} = x - undepressed phase advance

σ_x = x - depressed phase advance

to calculate the phace advance through axial "lattice period" **distance L_p for the** continuously focused beam. **Show that**

$$k_{\beta 0}^2 = \left(\frac{\sigma_0}{L_p}\right)^2$$

$$k_\beta^2 \equiv \left(\frac{\sigma}{L_p}\right)^2 = k_{\beta 0}^2 - \frac{Q}{r_b^2} = k_{\beta 0}^2 - \frac{\hat{\omega}_p^2}{2\gamma_b^3 \beta_b^2 c^2}$$

$$\hat{\omega}_p^2 \equiv \frac{q^2 \hat{n}}{m\varepsilon_0} = \text{plasma frequency squared.}$$

Problem 5, 25 Points

TED Problem 6

6/ For continuous focusing equilibria, it was shown that:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0 \gamma_b^3 \beta_b^2 c^2}\int_{\psi(r)}^{\infty} dH_\perp\, f_\perp(H_\perp)$$

$$\psi(r=0) = 0$$

a) Apply this formula to the thermal equilibrium distribution

$$f_\perp = \frac{\gamma_b m \beta_b^2 c^2 \hat{n}}{2\pi T}\exp\left\{-\frac{\gamma_b m \beta_b^2 c^2 H_\perp}{T}\right\}$$

to derive the transformed thermal equilibrium Poisson equation presented in class:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho\frac{\partial \tilde{\psi}}{\partial \rho}\right] = 1 + \Delta - e^{-\tilde{\psi}}$$

b) Show that the thermal equilibrium distribution satisfies the Density Inversion Theorem:

$$f_\perp(H_\perp) = -\frac{1}{2\pi}\left.\frac{\partial n}{\partial \psi}\right|_{\psi = H_\perp}$$

c) Verify the thermal equilibrium formula:

$$\varepsilon_x^2 = 16\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right] = \frac{16T}{\gamma_b m \beta_b^2 c^2}\langle x^2\rangle_\perp = \frac{4T}{\gamma_b m \beta_b^2 c^2} r_b^2$$

Hint for $\alpha > 0$:

$$\int_{-\infty}^{\infty} dx\; e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$

Take $\partial/\partial\alpha$ for other needed formulas.

Problem 6, 15 Points

TED Problem 5

S.M. Lund

5/ For a continuous focusing channel with

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$$

$$\varepsilon_x = \varepsilon_y = \text{const}$$

Consider, a round, "matched" KV equilibrium beam with

$$H_\perp = \frac{1}{2}(x'^2 + y'^2) + \frac{\varepsilon_x^2}{2 r_b^4}(x^2 + y^2) \qquad \text{Hamiltonian}$$

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0 \qquad \text{Envelope Eqn.}$$

Show that the KV equilibrium distribution

$$f_\perp = \frac{\hat{n}}{2\pi} \delta[H_\perp - H_b] \quad ; \quad H_b = \frac{\varepsilon_x^2}{2 r_b^2}$$

$$\hat{n} = \frac{\lambda}{\pi r_b^2} = \text{const}$$

yields

$$n(r) = \int d^2 x_\perp' \, f_\perp = \begin{cases} \hat{n} & ; \; r < r_b \\ 0 & ; \; r > r_b \end{cases}$$

Hints:

1) See steps carried out in Appendix B for an elliptical KV beam. These can be and/or applied more simply to the round beam.

2) See comments in notes on angular integrations

$$\int d^2 x_\perp' \ldots = \int dx' \int dy' \ldots \quad \text{with cylindrical symmetry}$$