Problem Set #5
NE 290H Barnard and Lund
Due March 2, 2009

S. M. Lund

P1

Problem #1
20 pts

TED Problem 1

1/ Consider a $\perp$ unbunched ion beam described by

$f_\perp(\vec{x}_\perp, \vec{x}_\perp{}', s)$ ~ single particle distribution. Satisfying Vlasov's equation.

$$H_\perp = \frac{1}{2}\vec{x}_\perp'^2 + \frac{\kappa_x(s)}{2}x^2 + \frac{\kappa_y(s)}{2}y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

$$\nabla_\perp^2\phi = \frac{-q}{\varepsilon_0}\int d^2x'\, f(\vec{x}_\perp, \vec{x}_\perp{}', s)$$

$\phi(r = r_p) = 0$ Grounded pipe boundary condition. r_p = pipe radius.

a) What are the first-order particle equations of motion for $d/ds\, \vec{x}_\perp$ and $\frac{d}{ds}\vec{x}_\perp{}'$ derived from $H_\perp$?

b) Using the results of part a), What is the 2nd-order particle equation of motion for $\frac{d^2}{ds^2}\vec{x}_\perp$?

c) Use the particle equations of motion to calculate d/ds of the single-particle Hamiltonian $H_\perp$ and the "angular momentum"

$$P_\theta \equiv xy' - yx'.$$

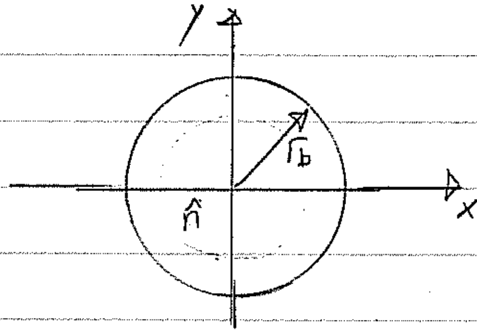
I.e., $\frac{d}{ds}H_\perp = ?$, $\frac{d}{ds}P_\theta = ?$

d) Use the expressions of part c) to show that for $\kappa_x = \text{const}$, $\kappa_y = \text{const}$, and $f_\perp = f_\perp(H_\perp)$ that $H_\perp = \text{const}$. Here, $f(H_\perp)$ can be any function of $H_\perp$ with $f(H_\perp) \geq 0$.

e) Use the expressions of part c) to show that for axisymmetric beams ($\partial/\partial\theta = 0$) with $\kappa_x = \kappa_y = \kappa(s)$ and $f_\perp = f_\perp(H_\perp)$ that $P_\theta = \text{const}$. θ = azimuthal angle

TED Problem 2

2/ Consider a uniform density beam in free-space with circular cross-section, edge radius r_b, and uniform in z $(\partial/\partial z = 0)$.



r_b = beam edge radius.

$r = \sqrt{x^2+y^2}$

$\hat{n}$ = const.

$\lambda = q\hat{n}\pi r_b^2$ = line-charge

a) Construct the solution to Poisson's equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{-q}{\epsilon_0}\begin{cases} \hat{n}, & r < r_b \\ 0, & r > r_b. \end{cases}$$

satisfying

$$\frac{\partial \phi}{\partial \theta} = 0 \quad \text{and} \quad \lim_{r\to\infty} -\frac{\partial \phi}{\partial r} = \frac{\lambda}{2\pi\epsilon_0 r}$$

b) Take derivatives of the interior solution $(r < r_b)$ in part a) to obtain formulas for

$$E_x = -\frac{\partial \phi}{\partial x}$$

$$E_y = -\frac{\partial \phi}{\partial y}$$

c) Show that the ellipsoidal beam formulas

$$E_x = -\frac{\partial \phi}{\partial x} = \frac{\lambda}{\pi\epsilon_0}\frac{x/r_x}{r_x + r_y}$$

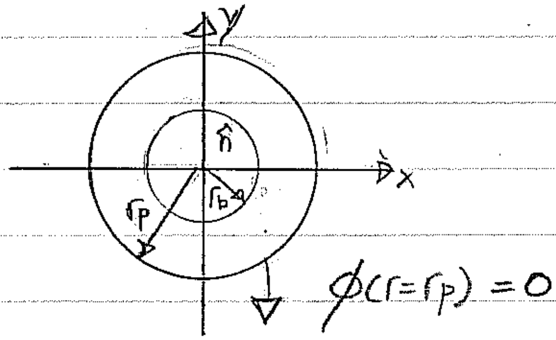
$$E_y = -\frac{\partial \phi}{\partial y} = \frac{\lambda}{\pi\epsilon_0}\frac{y/r_y}{r_x + r_y}$$

reduce to the results in part c) for a round beam with $r_x = r_y = r_b$.

S. M. Lund P2a/

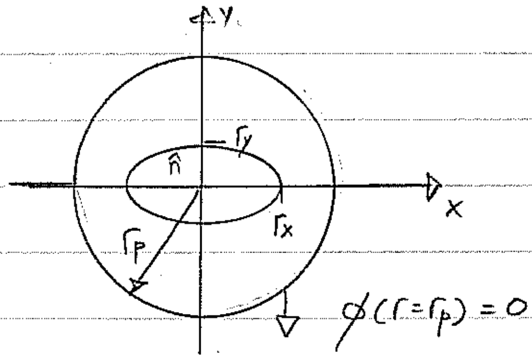
TED Problem 2

d) Would a grounded, conducting pipe of radius $r = r_p > r_b$ change the answers in part b)?



$r_p > r_b$

e) Would a grounded conducting pipe of radius $r = r_p > r_x, r_y$ change the fields calculated in class for the elliptical beam case with $r_x \neq r_y$? (no need to calculate any changes, just explain answer)



Problem #3 10 pts

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P3/

TED Problem 3.

3/ For a KV distribution:

$$n(x,y) = \int dx' dy' \, f_\perp = \begin{cases} \hat{n} \, , & \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1 \\ 0 \, ; & \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1 \end{cases}$$

Use this result to verify the formulas

$$r_x = 2\langle x^2 \rangle_\perp^{1/2}$$
$$r_y = 2\langle y^2 \rangle_\perp^{1/2}$$

Hint: Integrals may be more easily carried out if the elliptical integration domain is transformed to a circular domain. See for example, steps in Appendix A of the class notes.

TED Problem 4

Problem #4
10 pts

S.M. Lund

4/ For a continuous focusing channel with

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const.}$$

and a round, "matched" KV equilibrium beam with

$$\varepsilon_x = \varepsilon_y$$

$$r_x = r_y = r_b = \text{const}$$

a) Solve the envelope equation for the beam radius r_b in terms of the Perveance Q, $k_{\beta 0}$, and ε_x.

b) Solve for the zero space-charge amplitude function

$$W_0 = W_{0x} = W_{0y}$$

c) Apply the general phase advance formulas to calculate

$$\sigma_{0x} = x\text{- undepressed phase advance}$$

$$\sigma_x = x\text{- depressed phase advance}$$

to calculate the phace advance through a "lattice period" L_p. Show that

$$k_{\beta 0}^2 = \left(\frac{\sigma_0}{L_p}\right)^2$$

$$k_\beta^2 \equiv \left(\frac{\sigma}{L_p}\right)^2 = k_{\beta 0}^2 - \frac{Q}{r_b^2} = k_{\beta 0}^2 - \frac{\hat{\omega}_p^2}{2\gamma_b^3 \beta_b^2 c^2}$$

$$\hat{\omega}_p^2 = \frac{q^2 \hat{n}}{m\varepsilon_0} = \text{plasma frequency squared.}$$

Problem #5
15 pts

TED Problem 5

S.M. Lond

5/ For a continuous focusing channel with

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$$

$$\varepsilon_x = \varepsilon_y = \text{const}$$

Consider a round, "matched" KV equilibrium beam with

$$H_\perp = \frac{1}{2}(x'^2 + y'^2) + \frac{\varepsilon_x^2}{2 r_b^4}(x^2 + y^2) \quad \text{Hamiltonian}$$

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_x^2}{r_b^3} = 0 \quad \text{Envelope Eqn.}$$

Show that the KV equilibrium distribution

$$f_\perp = \frac{\hat{n}}{2\pi}\delta[H_\perp - H_b] \;; \quad H_b = \frac{\varepsilon_x^2}{2 r_b^2}$$

$$\hat{n} = \frac{\lambda}{\pi r_b^2} = \text{const}$$

yields

$$n(r) = \int d^2x_\perp' \, f_\perp = \begin{cases} \hat{n} \;; & r < r_b \\ 0 \;; & r > r_b \end{cases}$$

Hint:

See steps carried out in Appendix B for an KV