Problem Set # 3
NE290H Barnard and Lund
Due Feb. 11, 2009

Problem # 1
15 Points

S. M. Lund

JPE Problem 1 – Larmor Frame

1/ For a uniform solenoidal channel:

$$B_z^a(s) = B_0 = \text{const}$$

with no acceleration

$$\gamma_b \beta_b = \text{const}$$

and an axisymmetric $(\partial/\partial\theta = 0)$ beam with

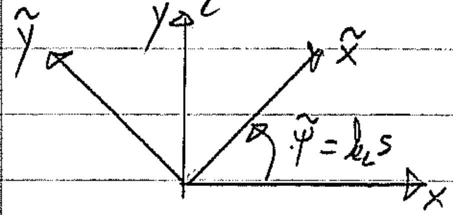
$$\frac{\partial\phi}{\partial \vec{x}_\perp} = \frac{\partial\phi}{\partial r}\frac{\partial r}{\partial \vec{x}_\perp} = \frac{\partial\phi}{\partial r}\frac{\vec{x}_\perp}{r} \qquad r = \sqrt{x^2+y^2}$$

the particle equations of motion reduce to:

$$x'' = \frac{qB_0}{m\gamma_b\beta_b c}\,y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{x}{r}$$

$$y'' = -\frac{qB_0}{m\gamma_b\beta_b c}\,x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{y}{r}$$

a) Parallel steps taken in the class notes to transform the equations of motion to a co-rotating frame:



$k_L = \text{const}$ = Larmor wavenumber

$$\tilde{x} = x\cos(k_L s) + y\sin(k_L s)$$
$$\tilde{y} = -x\sin(k_L s) + y\cos(k_L s)$$

Find an expression for k_L to reduce the equations of motion to the decoupled form:

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$$\tilde{x}'' + \kappa \tilde{x} = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r}$$

$$\tilde{y}'' + \kappa \tilde{y} = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r}$$

and identify $\kappa = \text{const.}$

Hint:
The transformation can be carried out directly. But you may find the algebra simpler using complex coordinates as in the class notes:

$$z = x + iy \qquad i = \sqrt{-1} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$
$$\tilde{z} = \tilde{x} + i\tilde{y}$$

b) If the direction of the magnetic field is reversed:
$$B_0 \rightarrow -B_0$$
how will the dynamics be influenced?

c) Neglect space-charge:
$$\phi = 0$$
and sketch a typical orbit in the rotating Larmor frame. Will this orbit appear more complicated in the Laboratory Frame? Why?

Bonus: Sketch the orbit taking advantage of simple choices of initial conditions that can always be made through choice of coordinates.

TPE Problem 2

Problem #2
10 Points

S.M. Lund P2/

2/ Derive the 2×2 transfer matrices $M(s|s_i)$ advancing the particle orbit $x(s)$, $x'(s)$ from the initial conditions at $s = s_i$

$$x(s_i) \equiv x_i$$
$$x'(s_i) \equiv x_i'$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M(s|s_i) \begin{pmatrix} x_i \\ x_i' \end{pmatrix}$$

for the following:

a) Free Drift

equation: $x'' = 0$

b) Continuous focusing:

equation: $x'' + k_{\beta 0}^2 x = 0$ $\qquad k_{\beta 0}^2 = \text{const.} > 0.$

c) Quadrupole focusing:

equation: $x'' + \kappa_q x = 0$ $\qquad \kappa_q = \text{const} > 0$

d) Quadrupole defocusing:

equation: $x'' - \kappa_q x = 0$ $\qquad \kappa_q = \text{const} > 0$

e) Verify in cases a) – d) that the Wronskian W satisfies:

$$W = \det M(s|s_i) = 1$$

TPE Problem 3

Problem #3
15 Points

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Thin Lens transfer Matrix for a single particle.

3/

Consider $\kappa_x = \pm \frac{1}{f}\delta(s-s_0)$ and the equation of motion:

$$x'' + \frac{1}{f}\delta(s-s_0)\,x = 0$$

$f = \text{const}$ (focal-length)

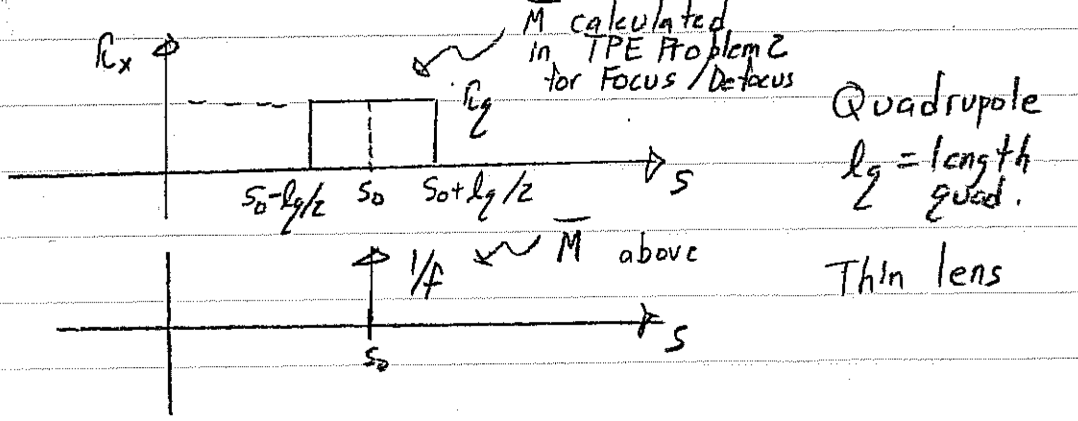
s_0 = axial location of the optic

a) Derive the 2x2 transfer matrix $\bar{M}$ for the optic:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0^+} = \bar{M} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0^-}$$

where $s_0^\pm$ are the coordinates infintesimally to the left and right of the optic. $\bar{M}$ is the so-called thin-lens tranfer matrix.

b) Construct a thin lens limit for 2x2 thick lens transfer matrices for focusing and defocusing quadropoles with $\kappa_q = \text{const}$ which obtains the same transfer matrix as in part a).



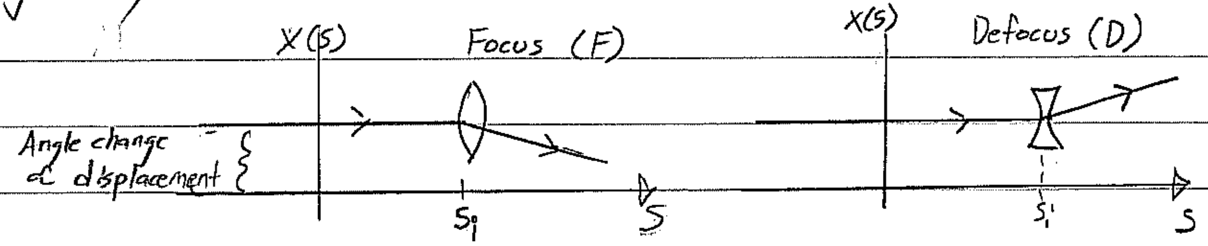
Hint: Require the same "impulse" $\int ds\, \kappa_x(s)$ to the particles to relate $\kappa_q l_q$ and f, then expand $\bar{M}$ for focus/defocus quad to get same $\bar{M}$ as in a).

TPE Problem 4

Problem #4
15 Points

S.M. Lund P4/

4/ A thin lense changes the angle of a particle trajectory but not the coordinate:



This action can be specified by transfer matrices applied at $s = s_i$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M(s|s_i)\begin{pmatrix} x_i \\ x_i' \end{pmatrix}$$

Focusing:

$$M_F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$f > 0$$

Defocusing:

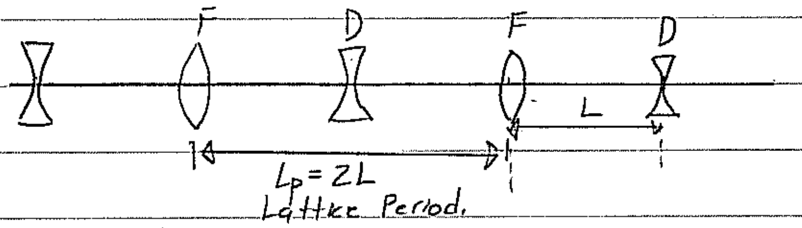
$$M_D = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$f > 0$$

From TPE Problem 2, a free-space drift of length L has a transport Matrix:

$$M_O = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Consider a lattice of period $2L$ made up of equally spaced F and D lenses with equal values of f.



This is the simplist "FODO" alternating gradient lattice!

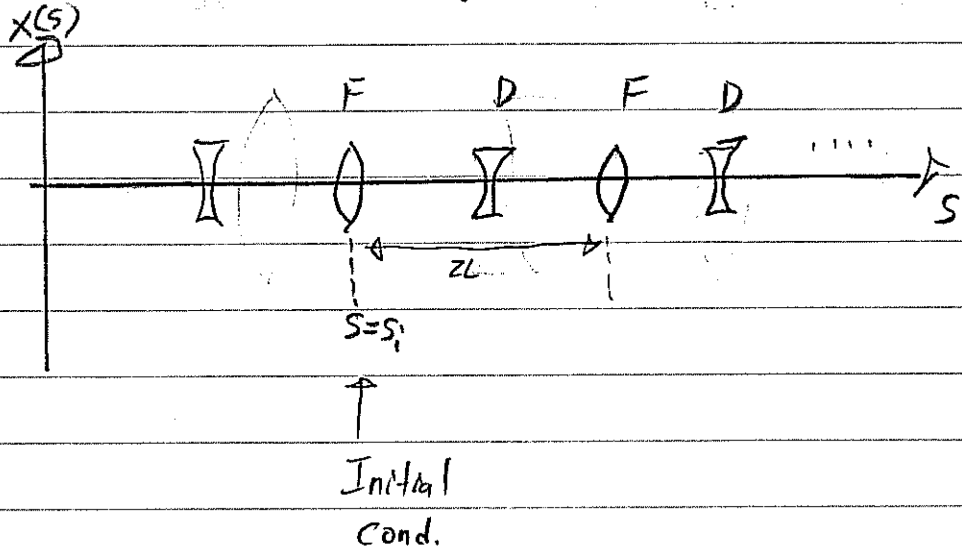
a) Use the transfer matrix analysis developed in class to find the range of f for which the particle orbit is stable.

b) Calculate $\cos\sigma_0$ where σ_0 is the particle phase advance.

c) For the case of of f chosen to correspond to the stability limit, sketch the motion of a particle with initial condition

$$\lim_{s \to s_i^-} x(s) = x_0$$
$$\lim_{s \to s_i^-} x'(s) = x_0/L$$

where $s = s_i$ is the axial location of a focusing thin lens kick, and $s \to s_i^-$ is just before the kick. Sketch the particle orbit for f slightly larger than the stability limit. Superimpose the orbit sketch on a diagram of the lattice (see below):



Problem #5
20 Points

TPE Problem 5

S.M. Lund P5/

5/ Consider a static field magnet with vacuum aperture field satisfying

$$\nabla \cdot \vec{B}^{q} = 0$$

$$\nabla \times \vec{B}^{q} = 0$$

Define 2D field components:

$$B_x(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\ B_x^{q}(x,y,z)$$

$$B_y(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\ B_y^{q}(x,y,z)$$

where $\ell = \text{const.}$ is any length scale.

a) Show that the 2D field components satisfy (finite magnet)

$$\left.\begin{aligned} \frac{\partial B_x(x,y)}{\partial x} &= -\frac{\partial B_y(x,y)}{\partial y} \\ \frac{\partial B_x(x,y)}{\partial y} &= \frac{\partial B_y(x,y)}{\partial x} \end{aligned}\right\}$$

Cauchy-Riemann conditions for $\underset{\sim}{B} = B_y + i B_x$ to be an analytical function of $\underset{\sim}{z} = x + iy$ $\quad i = \sqrt{-1}$

b) Write out 2D field components for normal and skew component quadrupole and sextupole magnets.

e.g. Normal quadrupole $n = 2,\ b_2 \neq 0,\ a_2 = 0$

Skew quadrupole $n = 2,\ b_2 = 0,\ a_2 \neq 0$, etc.

c) Using $\vec{F} = q\beta_b c\ \hat{z} \times \vec{B}_\perp$ $\quad \vec{B}_\perp = B_x\hat{x} + B_y\hat{y}$

sketch lines of force acting on a particle with charge q moving along the z-axis for a normal quadrupole and then a skew quadrupole. How do these differ?

d) Will the normal and skew terms be defined the same for electric quadrupole focusing or should they be interchanged?

i.e.,

$$\underset{\sim}{E} = E_y + i E_x = \sum_{n=1}^{\infty} \underset{\sim}{E_n} \left(\underset{\sim}{z} / r_p \right)^{n-1}$$

$$\underset{\sim}{E_n} \equiv \underset{\text{Skew}}{a_n} + i \underset{\text{Normal}}{b_n}$$

or

$$\underset{\sim}{E_n} \equiv \underset{\text{Normal}}{b_n} + i \underset{\text{Skew}}{a_n}$$

e) What ideal pole orientation is necessary to generate magnetic normal and skew quadrupole field components? Only sketches are necessary.