Transverse Particle Resonances with Application to Circular Accelerators*

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Transverse Particle Resonances: Outline

Transverse Particle Resonances: Detailed Outline

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Acknowledgments

S1: Overview

In our treatment of transverse single particle orbits of lattices with s-varying focusing, we found that Hill's Equation describes the orbits to leading-order approximation:

$$\boxed{\begin{aligned} x''(s) + \kappa_x(s)x(s) &= 0 \\ y''(s) + \kappa_y(s)y(s) &= 0 \end{aligned}}$$

where $\kappa_x(s), \quad \kappa_y(s)$

are functions that describe the linear applied focusing fields of the lattice

- Focusing functions can also incorporate linear space-charge forces
  - Self-consistent for special case of a KV distribution

In analyzing Hill's equations we employed phase-amplitude methods

- See: S.M. Lund lectures on Transverse Particle Equations, S8, on the betatron form of the solution

$$x(s) = A_{xi}\sqrt{\beta_x(s)}\cos\psi_x(s) \qquad A_{xi} = \text{const}$$

$$\frac{1}{2}\beta_x(s)\beta_x''(s) - \frac{1}{4}\beta_x'^2(s) + \kappa_x(s)\beta_x^2(s) = 1 \qquad \psi_x(s) = \psi_{xi} + \int_{s_i}^{s} \frac{d\bar{s}}{\beta_x(\bar{s})}$$

$$\beta_x(s + L_p) = \beta_x(s)$$

This formulation helped to simply identify the Courant-Snyder invariant:

$$\boxed{\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_x^2 = \text{const}} \qquad w_x = \sqrt{\beta_x}$$

which helped to interpret the dynamics.

We will now exploit this formulation to better (analytically!) understand resonant instabilities in periodic focusing lattices. This is done by choosing coordinates such that stable unperturbed orbits described by Hill's equation:

$$\boxed{x''(s) + \kappa_x(s) x(s) = 0}$$

are mapped to a continuous oscillator

$$\boxed{\begin{gathered} \tilde{x}''(\tilde{s}) + \tilde{k}_{\beta 0}^2 \tilde{x}(\tilde{s}) = 0 \\ \tilde{k}_{\beta 0}^2 = \text{const} > 0 \end{gathered}} \qquad \cdot\tilde{\cdot}\cdot = \text{Transformed Coordinate}$$

These transforms will help us more simply understand the action of perturbations (from applied field nonlinearities, ...) acting on the particle orbits:

$$\boxed{\begin{aligned} x''(s) + \kappa_x(s)x(s) &= \mathcal{P}_x(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta}) \\ y''(s) + \kappa_y(s)y(s) &= \mathcal{P}_y(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta}) \\ \mathcal{P}_x,\ \mathcal{P}_y &= \text{Perturbations} \\ \vec{\delta} &= \text{Extra Coupling Variables} \end{aligned}}$$

For simplicity, we restrict analysis to:

$$\gamma_b\beta_b = \text{const} \qquad \text{No Acceleration}$$

$$\delta = 0 \qquad \text{No Axial Momentum Spread}$$

$$\phi = 0 \qquad \text{Neglect Space-Charge}$$

- Acceleration can be incorporated using transformations (see Transverse Particle Equations of Motion lectures)
- Comments on space-charge effects will be made in S7

We also take the applied focusing lattice to be periodic with:

$$\boxed{\begin{aligned} \kappa_x(s + L_p) &= \kappa_x(s) \\ \kappa_y(s + L_p) &= \kappa_y(s) \end{aligned} \qquad L_p = \text{Lattice Period}}$$

For a ring we also always have the superperiodicity condition:

$$\mathcal{P}_x(s + \mathcal{C}; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta}) = \mathcal{P}_x(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$

$$\mathcal{P}_y(s + \mathcal{C}; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta}) = \mathcal{P}_y(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$

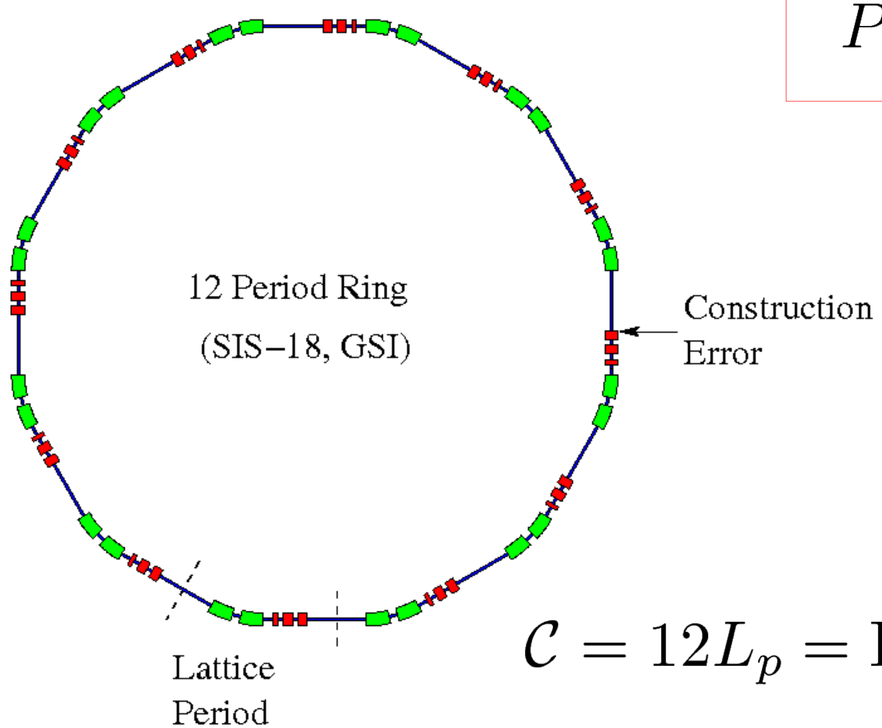
$$\mathcal{C} = \mathcal{N} L_p = \text{Circumference Ring}$$

$$\mathcal{N} \equiv \text{Superperiodicity}$$

Perturbations can be Random and/or Systematic:

Random Errors in a ring will be felt once per particle lap in the ring rather than every lattice period

$$P_{x,y}(\cdots, s + \mathcal{N} L_p) = P_{x,y}(\cdots, s)$$



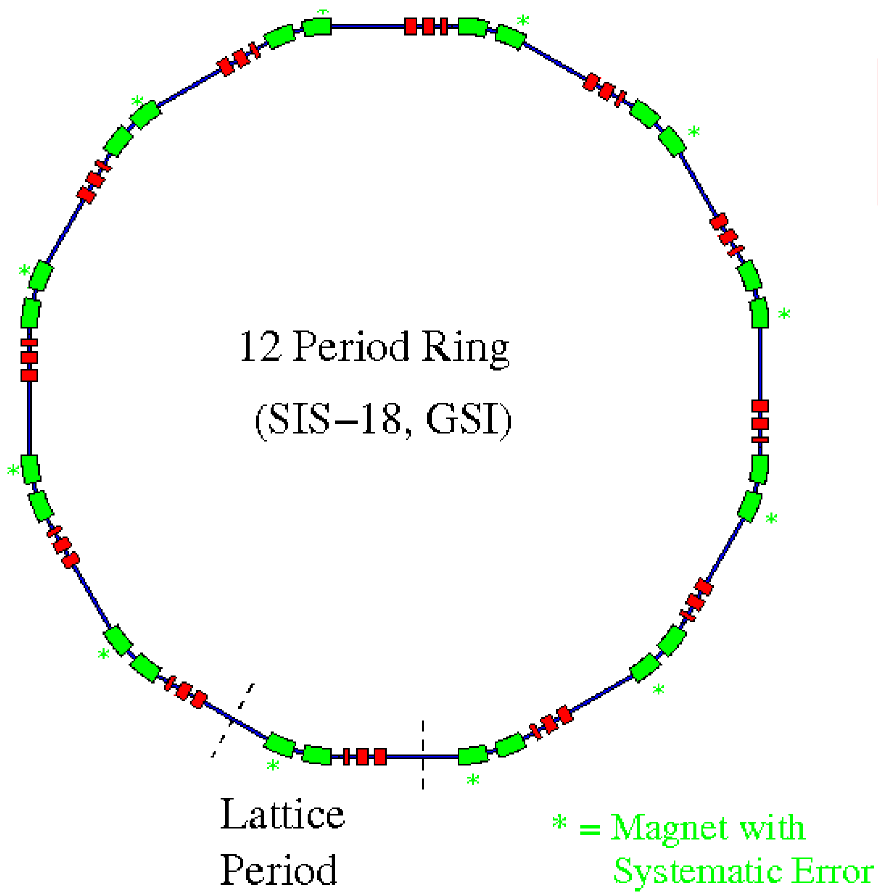
Random Error Sources:

- Fabrication
- Assembly/Construction
- Material Defects
- ....

$$\mathcal{C} = 12 L_p = \text{Ring Circumference}$$

Systematic Errors can occur in both linear machines and rings and effect *every* lattice period in the same manner.

Example: FODO Lattice with the same error in each dipole of pair



$$P_{x,y}(\cdots, s + L_p) = P_{x,y}(\cdots, s)$$

Systematic Error Sources:

- Design Flaw/Limit/Ideal
- Repeated Construction
- or Material Error
- ....

We will find that perturbations arising from both random and systematic error can drive resonance phenomena that destabilize particle orbits and limit machine performance

S2: Floquet Coordinates and Hill's Equation

Define for a *stable* solution to Hill's Equation

- Drop x subscripts and only analyze x-orbit for now to simplify

"Radial" Coordinate: $\quad u \equiv \frac{x}{\sqrt{\beta}}$
(dimensionless)

"Angle" Coordinate: $\quad \varphi \equiv \frac{1}{\nu_0}\int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \equiv \frac{\Delta\psi(s)}{\nu_0}$
(dimensionless, normalized)

where:

$$\beta = w^2 = \text{Betatron Amplitude Function}$$

$$\nu_0 \equiv \frac{\Delta\psi(\mathcal{N}L_p)}{2\pi} = \begin{array}{l}\text{Number undepressed particle oscillations} \\ \text{in ring } (\mathcal{N} = \text{Superperiod Number})\end{array}$$

$$\psi = \text{Phase of } x\text{-orbit}$$

$$\Delta\psi(s) = \psi(s) - \psi(s_i)$$

Can also take $\mathcal{N} = 1$ and then ν_0 is the number (usually fraction thereof) of undepressed particle oscillations in *one* lattice period

Comment:

φ can be interpreted as a normalized angle measured in the particle betatron phase advance:

Ring: $(\mathcal{N} = \text{Superperiod } \#)$ $\Longrightarrow$ φ advances by 2π on one transit around ring for analysis of Random Errors

Linac or Ring: $(\mathcal{N} = 1)$ $\Longrightarrow$ φ advances by 2π on transit through one lattice period for analysis of Systematic Errors in a ring *or* linac

Take φ as the independent coordinate:

$$\boxed{u = u(\varphi)}$$

and define a new "momentum" phase-space coordinate

$$\boxed{\dot{u} \equiv \frac{du}{d\varphi}} \qquad \boxed{\dot{} \equiv \frac{d}{d\varphi}}$$

These new variables will be applied to express Hill's equation in a simpler form

From the definition

$$u \equiv \frac{x}{\sqrt{\beta}}$$

Rearranging this and using the chain rule:

$$x = \sqrt{\beta} u$$

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \sqrt{\beta} \frac{du}{d\varphi} \frac{d\varphi}{ds} \qquad\qquad \frac{d}{ds} = \frac{d\varphi}{ds} \frac{d}{d\varphi}$$

From:

$$\varphi \equiv \frac{1}{\nu_0} \int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \qquad \Longrightarrow \qquad \boxed{\frac{d\varphi}{ds} = \frac{1}{\nu_0 \beta}}$$

we obtain

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \frac{1}{\nu_0 \sqrt{\beta}} \dot{u}$$

$$x'' = \frac{d}{ds} x' = \frac{\beta''}{2\sqrt{\beta}} u - \frac{\beta'^2}{4\beta^{3/2}} u + \overbrace{\cancel{\frac{\beta'}{2\nu_0 \beta^{3/2}} \dot{u}} - \cancel{\frac{\beta'}{2\nu_0 \beta^{3/2}} \dot{u}}}^{0 \;\; \text{(cancels)}} + \frac{1}{\nu_0^2 \beta^{3/2}} \ddot{u}$$

Summary:

$$x' = \frac{\beta'}{2\sqrt{\beta}}u + \frac{1}{\nu_0\sqrt{\beta}}\dot{u}$$

$$x'' = \frac{\beta''}{2\sqrt{\beta}}u - \frac{\beta'^2}{4\beta^{3/2}}u + \frac{1}{\nu_0^2\beta^{3/2}}\ddot{u}$$

Using these results, Hill's equation:

$$x''(s) + \kappa_x(s)x(s) = 0$$

becomes

$$\ddot{u} + \nu_0^2\left[\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + \kappa\beta^2\right]u = 0$$

But the betatron amplitude equation satisfies:

$$\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + \kappa\beta^2 = 1 \qquad \beta(s+L_p) = \beta(s)$$

Thus the terms in [...] = 1 and Hill's equation reduces to simple harmonic oscillator form:

$$\boxed{\ddot{u} + \nu_0^2 u = 0 \qquad \nu_0^2 = \text{const} > 0}$$

Transform has mapped a stable, time dependent solution to Hill's equation to a simple harmonic oscillator!

The general solution to the simple harmonic oscillator equation can be expressed as:

$$u(\varphi) = u_i \cos(\nu_0 \varphi) + \frac{\dot{u}_i}{\nu_0} \sin(\nu_0 \varphi)$$

$$\dot{u}(\varphi) = -u_i \nu_0 \sin(\nu_0 \varphi) + \dot{u}_i \cos(\nu_0 \varphi)$$

$$u(\varphi = 0) = u_i = \text{const}$$

$$\dot{u}(\varphi = 0) = \dot{u}_i = \text{const}$$

u_i and $\dot{u}_i$ set by x, x' initial conditions at $s = s_i$ (phase choice $\varphi = 0$ at $s = s_i$)

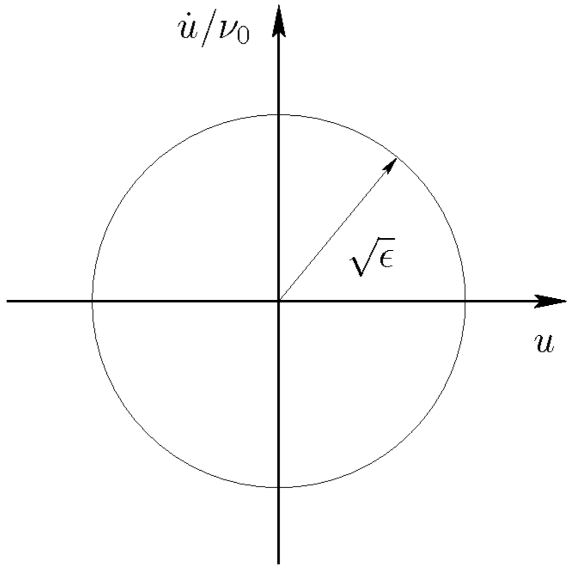
The Floquet representation also simplifies the interpretation of the Courant-Snyder invariant:

$$u^2 + \left(\frac{\dot{u}}{\nu_0}\right)^2 = u_i^2 + \left(\frac{\dot{u}_i}{\nu_0}\right)^2 \equiv \epsilon = \text{const}$$

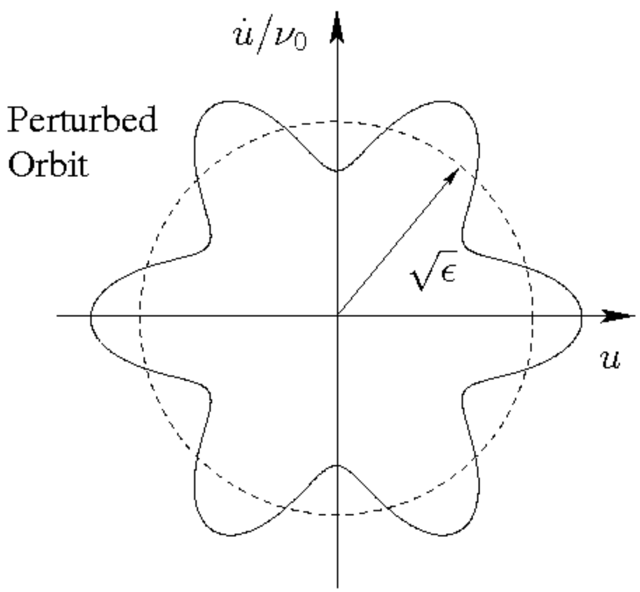
- Unperturbed phase-space in $u - \dot{u}/\nu_0$ is a unit circle of area $\pi\epsilon$!
- Relate this area to *x*-*x*' phase-space area shortly

  -- Preview: areas are equal

Unperturbed phase-space ellipse:



This simple structure will also allow more simple visualization of perturbations as distortions on a unit circle, thereby clarifying symmetries:



Poor example .... better to draw
a random perturbation acting once per
lap/turn .... will update plot.

The $u - \dot{u}/\nu_0$ variables also preserve phase-space area

- Feature of the transform being symplectic (Hamiltonian Dynamics)

From previous results:

$$x = \sqrt{\beta} u \qquad \frac{d\varphi}{ds} = \frac{1}{\nu_0 \beta}$$

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \sqrt{\beta} \frac{d\varphi}{ds} \dot{u} = \frac{\beta'}{2\sqrt{\beta}} u + \frac{1}{\nu_0 \sqrt{\beta}} \dot{u}$$

Transform area elements by calculating the Jacobian:

$$dx \otimes dx' = |J| du \otimes d\dot{u}$$

$$J = \det \left| \begin{array}{cc} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial \dot{u}}, \\ \frac{\partial x'}{\partial u} & \frac{\partial x'}{\partial \dot{u}} \end{array} \right] = \det \left| \begin{array}{cc} \sqrt{\beta} & 0 \\ \frac{\beta'}{2\sqrt{\beta}} & \frac{1}{\nu_0 \sqrt{\beta}} \end{array} \right] = \frac{1}{\nu_0}$$

$$\boxed{dx \otimes dx' = du \otimes \frac{d\dot{u}}{\nu_0}}$$

Thus the Courant-Snyder invariant ϵ is the usual single particle emittance in *x*-*x*' phase-space

S3: Perturbed Hill's Equation in Floquet Coordinates

Return to the perturbed Hill's equation in S1:

$$\boxed{\begin{aligned} x''(s) + \kappa_x(s)x(s) &= \mathcal{P}_x(s;\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta}) \\ y''(s) + \kappa_y(s)y(s) &= \mathcal{P}_y(s;\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta}) \\ \mathcal{P}_x,\ \mathcal{P}_y &= \text{Perturbations} \\ \vec{\delta} &= \text{Extra Coupling Variables} \end{aligned}}$$

Drop the extra coupling variables and apply the Floquet transform in S2:

- Examine only *x*-equation, *y*-equation analogous
- Drop *x*-subscript in $\mathcal{P}_x$ to simplify notation

$$\boxed{\ddot{u} + \nu_0^2 u = \nu_0^2 \beta^{3/2}\mathcal{P}}$$

Here,

$$\mathcal{P} = \mathcal{P}(s(\varphi), \sqrt{\beta}u, y, \vec{\delta})$$

Transform *y* similarly to *x*

Expand the perturbation in a power series:

- Can be done for all physical applied field perturbations
- Multipole symmetries can restrict the form of the perturbations
  - See: S4 in these notes and S3 in Transverse Particle Equations of Motion
- Perturbations can be random (once per lap) or systematic (every lattice period)

$$\begin{aligned}\mathcal{P}(x,y,s) &= \mathcal{P}_0(y,s) + \mathcal{P}_1(y,s)x + \mathcal{P}_2(y,s)x^2 + \cdots \\ &= \sum_{n=0}^{\infty} \mathcal{P}_n(y,s)x^n\end{aligned}$$

Take:

$$x = \sqrt{\beta}u$$

to obtain:

$$\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} \mathcal{P}_n(y,s) u^n$$

A similar equation applies in the *y*-plane.

S4: Sources and Forms of Perturbation Terms

Within a 2D transverse model it was shown that applied magnetic fields can be expanded as:

- See: S3, Transverse Particle Equations of Motion
- Applied electric fields can be analogously expanded

$$\underline{B}^a(\underline{z}) = B_y^a + iB_x^a = \sum_{n=1}^{\infty} \underline{B}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1}$$

$$\underline{B}_n = \text{const (complex)} \equiv b_n + ia_n \qquad \underline{z} = x + iy \qquad i = \sqrt{-1}$$

$$n = \text{Multipole Index} \qquad r_p = \text{Aperture "Pipe" Radius}$$

Index n	Name	Normal Field Components $B_x^a r_p^{n-1}/b_n$	Normal Field Components $B_y^a r_p^{n-1}/b_n$	Skew Field Components $B_x^a r_p^{n-1}/a_n$	Skew Field Components $B_y^a r_p^{n-1}/a_n$
$n=1$	Dipole	0	1	1	0
$n=2$	Quadrupole	y	x	x	$-y$
$n=3$	Sextupole	$2xy$	x^2-y^2	x^2-y^2	$-2xy$
$n=4$	Octupole	$3x^2y-y^3$	x^3-3xy^2	x^3-3xy^2	$-3x^2y+y^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Trace back how the applied magnetic field terms enter the *x*-plane equation of motion:

- See: S2, Transverse Particle Equations of Motion
- Apply equation in S2 with: $\beta_b = \text{const}, \;\; \phi \simeq \text{const}, \;\; E_x^a \simeq 0, \;\; B_z^a \simeq 0$

$$\boxed{x'' = -\frac{q}{m\gamma_b\beta_b c} B_y^a}$$

Apply this reduced equation with:

- Part of applied magnetic field giving linear focusing contained in κ

$$x'' + \kappa_x(s)x = -\frac{q}{m\gamma_b\beta_b c} B_y^a(x, y, s)$$

Nonlinear focusing terms only

Compare to form of perturbed Hill's equation:

$$x'' + \kappa_x x = \mathcal{P}_x = \sum_{n=0}^{\infty} \mathcal{P}_n(y, s)x^n \qquad \Longrightarrow \qquad \boxed{\mathcal{P}_x = -\frac{q}{m\gamma_b\beta_b c} B_y^a}$$

- Similar steps to identify *y*-plane perturbation terms or Electric perturbations
- Caution: Multipole index *n* and power series index *n* not always the same
  - In *x*-plane motion with *y* = 0 = *y*' (case illustrated later) they are the same

S6: Solution of the Perturbed Hill's Equation: Resonances

We will analyze the solution of the perturbed orbit equation:

$$\boxed{\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} \mathcal{P}_n(y,s) u^n}$$

derived in S4.

To more simply illustrate resonances, we analyze motion in the x-plane with:

$$\boxed{y(s) \equiv 0}$$

- Essential character of general analysis illustrated more simply
- Could generalize by expanding $\mathcal{P}_n(y,s)$ in a power series in y and generalizing notation to distinguish between Floquet coordinates in the *x*- and *y*-planes
  - Results in coupled *x* and *y*-equations of motion

Note that each *n*-labeled perturbation expansion coefficient is periodic with period of the ring circumference (random perturbations) or lattice period (systematic):

$$\beta(s+L_p) = \beta(s), \quad \mathcal{P}_n(y, s+\mathcal{N}L_p) = \mathcal{P}_n(y,s)$$

$$\Longrightarrow \quad \beta^{\frac{n+3}{2}}(s+\mathcal{N}L_p)\mathcal{P}_n(y,s+\mathcal{N}L_p) = \beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y,s)$$

Expand each *n*-labeled perturbation expansion coefficient in a Fourier series as:

$$\beta^{\frac{n+3}{2}}\mathcal{P}_n(y=0,s) = \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ikp\varphi}$$

$$i \equiv \sqrt{-1} \qquad p \equiv \begin{cases} 1, & \text{Random perturbation} \\ & \text{(once per lap in ring)} \\ & \\ \mathcal{N}, & \text{Systematic perturbation} \\ & \text{(every lattice period)} \end{cases}$$

$$C_{n,k} = \int_{-\pi/p}^{\pi/p} \frac{d\varphi}{2\pi}\, e^{-ikp\varphi}\beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y=0,s) \quad = \quad \text{const}$$

$$s = s(\varphi) \qquad \varphi = \int_{s_0}^{s} \frac{1}{\nu_0}\frac{d\tilde{s}}{\beta(\tilde{s})}$$

- Can apply to Rings for periodic perturbations (with $p = 1$) or random perturbations (with $p = N$)
- Can apply to linacs for periodic perturbations (every lattice period) with $p = 1$
- Does not apply to random perturbations in a linac

The perturbed equation of motion becomes:

$$\boxed{\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ikp\varphi} u^n}$$

Expand the solution as:

$$\boxed{u = u_0 + \delta u}$$

$$u_0 = \text{unpreturbed solution}$$

$$\delta u = \text{perturbation due to errors}$$

where u_0 is the solution to the *simple harmonic oscillator* equation in the absence of perturbations:

$$\boxed{\ddot{u}_0 + \nu_0^2 u_0 = 0}$$

Unperturbed equation of motion

Assume small-amplitude perturbations so that

$$|u_0| \gg |\delta u|$$

Then to linear order, the equation of motion for δu is:

$$\boxed{\ddot{\delta u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ipk\varphi} u_0^n}$$

Perturbed equation of motion

To obtain the perturbed equation of motion, the unperturbed solution u_0 is inserted on the RHS terms

- Gives simple harmonic oscillator equation with driving terms

Solution of the unpreturbed orbit is simply expressed as:

$$u_0 = u_{0i} \cos(\nu_0 \varphi + \psi_i)$$

$$u_{0i} = \text{const}$$

$$\psi_i = \text{const}$$

Set by particle initial conditions

Then binomial expand:

$$\begin{aligned} u_0^n &= u_{0i}^n \left(\frac{e^{i(\nu_0\varphi + \psi_i)} + e^{-i(\nu_0\varphi+\psi_i)}}{2} \right)^n \\ &= \frac{u_{oi}^n}{2^n} \sum_{m=0}^{n} \binom{n}{m} e^{i(n-m)(\nu_0\varphi+\psi_i)} e^{-im(\nu_0\varphi+\psi_i)} \\ &= \frac{u_{oi}^n}{2^n} \sum_{m=0}^{n} \binom{n}{m} e^{i(n-2m)\nu_0\varphi} e^{i(n-2m)\psi_i} \end{aligned}$$

where $\binom{n}{m} \equiv \dfrac{n!}{m!(n-m)!}$ is a binomial coefficient

Using this expansion the linearized perturbed equation of motion becomes:

$$\ddot{\delta u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} \sum_{m=0}^{n} \binom{n}{m} C_{n,k} e^{i[(n-2m)\nu_0 + pk]\varphi} e^{i(n-2m)\psi_i}$$

The solution for δu can be expanded as:

$$\delta u = \delta u_h + \delta u_p$$

$$\delta u_h = \text{homogenous solution}$$

General solution to: $\ddot{\delta u}_h + \nu_0^2 \delta u_h = 0$

$$\delta u_h = \text{particular solution}$$

Any solution with: $\delta u \rightarrow \delta u_p$

- Can drop homogeneous solution because it can be absorbed in unperturbed solution u_0
- Only a particular solution need be found

$$\delta u = \delta u_p$$

$$\boxed{\delta\ddot{u} + \nu_0^2\delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} \sum_{m=0}^{n} \binom{n}{m} C_{n,k} e^{i[(n-2m)\nu_0+pk]\varphi} e^{i(n-2m)\psi_i}}$$

Equation describes a driven simple harmonic oscillator with a periodic driving terms on the RHS:

- Homework problem reviews that solution of such an equation will be unstable when the driving term has a frequency component equal to the restoring term
  - Resonant exchange and amplitude grows *linearly* in φ
  - Parameters meeting resonance condition will lead to instabilities

Resonances occur when:

$$\boxed{(n-2m)\nu_0 + pk = \pm\nu_0}$$

is satisfied for the operating tune ν_0 and some values of:

$$n = 1,\ 2,\ 3,\ \cdots \qquad m = 0,\ 1,\ 2,\ \cdots,\ n$$

$$k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty$$

$$p \equiv \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

If growth rate is sufficiently large, machine operating points satisfying the resonance condition will be problematic since particles will be lost (scraped) by the machine aperture:

Machine operating tune (ν_0) can be moved to avoid

Perturbation can be actively corrected to reduce amplitude of driving term

Low order resonance terms with smaller *n*, *k*, *m* magnitudes are expected to be more dangerous because:

- Less likely to be washed out by effects not included in model
- Amplitude coefficients expected to be stronger

In the next section we will examine how resonances restrict possible machine operating parameters.

S7: Machine Operating Points: Tune Restrictions Resulting from Resonances

Examine situations where the resonance condition:

$$\boxed{(n - 2m)\nu_0 + pk = \pm\nu_0}$$

is satisfied for the operating tune ν_0 and some values of:

$$n = 1,\ 2,\ 3,\ \cdots \qquad m = 0,\ 1,\ 2,\ \cdots,\ n$$

$$k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty$$

$$p \equiv \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

Resonances can be analyzed one at a time using linear superposition

- Analysis valid for small-amplitudes

Analyze resonance possibilities starting with index *n* <==> Multipole Order

$n = 0$, Dipole Perturbations:

$$n = 0, \quad \Longrightarrow \quad m = 0$$

and the resonance condition gives:

$$\nu_0 = \pm pk \qquad\qquad pk = \text{integer} \qquad\qquad k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$$

$$p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

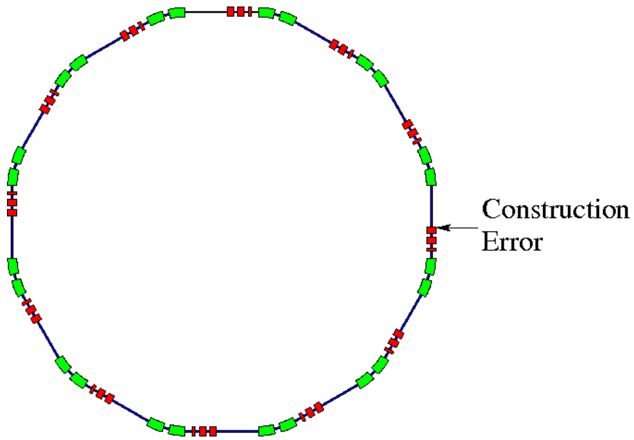
Therefore, to avoid dipole resonances:

$p = 1$	Random Perturbation	$\nu_0 \neq 1, 2, 3, \cdots$
$p = \mathcal{N}$	Systematic Perturbation	$\nu_0 \neq \mathcal{N}, 2\mathcal{N}, 3\mathcal{N}, \cdots$

- Systematic errors are less restrictive on machine operating points
- Multiply random perturbation tune restrictions by $\mathcal{N}$ to obtain the systematic perturbation case

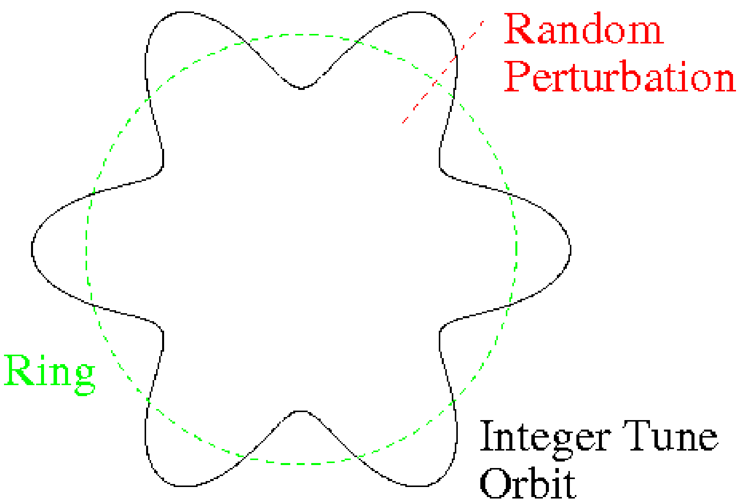
Interpretation of result:

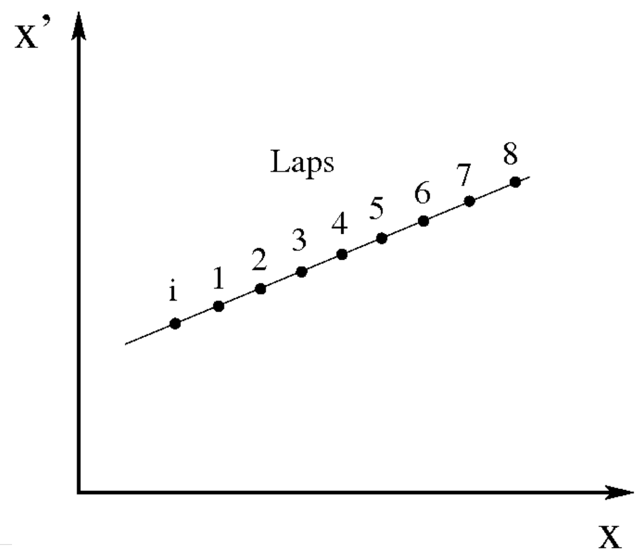
Consider a ring with a single (random) dipole error along the reference path of the ring:



If the particle is oscillating with integer tune, then the particle experiences the dipole error on each lap in the same oscillation phase and the trajectory will "walk-off" on a lap-to-lap basis in phase-space:

- With finite machine aperture the particle will be scraped/lost





$n = 1$, Quadrupole Perturbations:

$$n = 1, \implies m = 0,\ 1$$

and the resonance conditions give:

$$\begin{aligned} n = 1,\ m = 0: &\qquad \nu_0 + pk = \pm\nu_0 \\ n = 1,\ m = 1: &\quad -\nu_0 + pk = \pm\nu_0 \end{aligned} \implies pk = 0,\ \nu_0 = \pm\frac{pk}{2}$$

Implications:

$$1)\ pk = 0 \ \Rightarrow k = 0$$

Case can be treated by "renormalizing" oscillator focusing strength and need not be considered

$$\ddot{u} + \nu_0^2 u = \nu_0^2 C_{1,0} u$$

$$2)\ \nu_0 = \pm\frac{pk}{2} \ \Rightarrow \nu_0 = \frac{|pk|}{2}$$

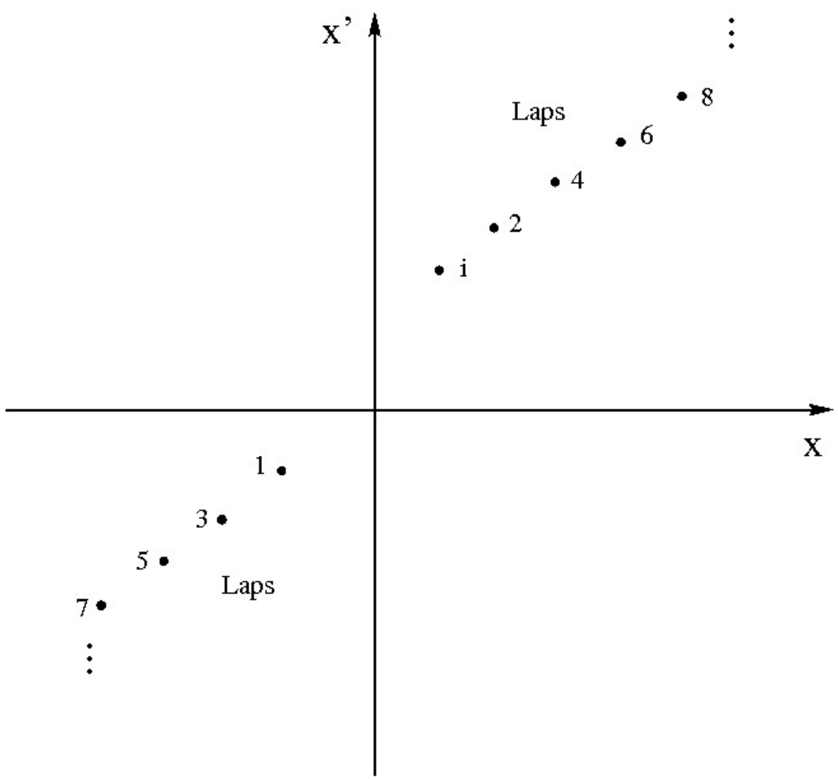
Therefore, to avoid quadrupole resonances:

$$\boxed{\nu_0 \neq \frac{|pk|}{2} \qquad p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases} \quad k = -\infty, \cdots, -1, 0, 1, \cdots, \infty}$$

- New restriction on machine tunes from being half-integer values
- Integers also restricted (some for $p = N$), but redundant with dipole case!

Interpretation of result (new restrictions):

For a single (random) quadrupole error along the azimuth of a ring, a similar qualitative argument as presented in the dipole resonance case leads on to conclude that if a particle oscillates with ½ integer tune, then the orbit can "walk-off" on a lap-to-lap basis in phase-space:



$n = 2$, Sextupole Perturbations:

$$n = 2, \quad \Longrightarrow \quad m = 0,\ 1,\ 2$$

and the resonance conditions give:

$$n = 2,\ m = 0: \qquad 2\nu_0 + pk = \pm\nu_0$$

$$n = 2,\ m = 1: \qquad pk = \pm\nu_0$$

$$n = 2,\ m = 2: \qquad -2\nu_0 + pk = \pm\nu_0$$

Therefore, to avoid sextupole resonances:

$$\boxed{\nu_0 \neq \begin{cases} |pk| & \text{integer} \\ |pk|/2 & \text{half-integer} \\ |pk|/3 & \text{third-integer} \end{cases} \qquad p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases} \qquad k = -\infty, \cdots, -1, 0, 1, \cdots, \infty}$$

- Integer and ½-integer restrictions already obtained for dipole and quadrupole perturbations
- 1/3-integer restriction new

Higher-order ($n > 2$) cases analyzed analogously

General form of resonances

The general resonance condition (all *n*-values) for *x*-plane motion can be summarized as:

$$M\nu_0 = N \qquad \begin{aligned} M,\ N &= \text{Integers of same sign} \\ M &= \text{"Order" of resonance} \end{aligned}$$

- Higher order numbers *M* are generally less dangerous
  - Longer coherence length for validity of theory: effects not included can "wash-out" the resonance
  - Coefficients generally smaller

Particle motion is not, in general, restricted to the *x*-plane, and a more general analysis taking into account coupled *x*- and *y*-plane motion shows that the generalized resonance condition is:

$$M_x\nu_{0x} + M_y\nu_{0y} = N \qquad \begin{aligned} M_x,\ M_y,\ N &= \text{Integers of same sign} \\ |M_x| + |M_y| &= \text{"Order" of resonance} \end{aligned}$$

$$\begin{aligned} \nu_{0x} &= x\text{-plane tune} \\ \nu_{0y} &= y\text{-plane tune} \end{aligned}$$

- Lower order resonances are generally more dangerous analogously to *x*-case

Restrictions on machine operating points

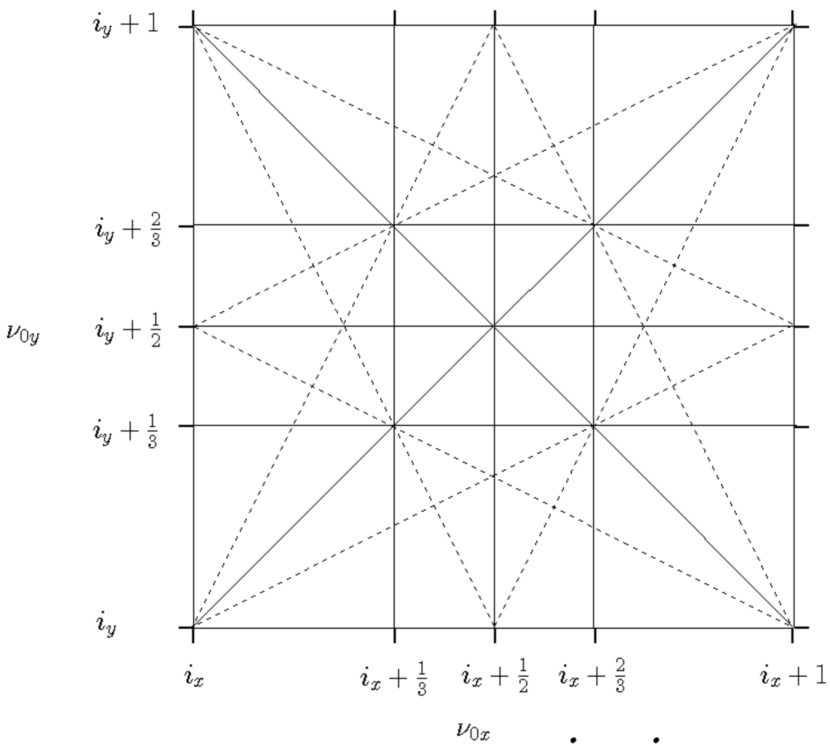
Tune restrictions are generally plotted in $\nu_{0x} - \nu_{0y}$ space order-by-order up to a max order value to find allowed tunes where the machine can safely operate

- Often 3rd order is chosen as a maximum to consider
- Cases for random (p = 1) and systematic (p = N) perturbations considered

Machine operating points chosen as far as possible from low order resonance lines

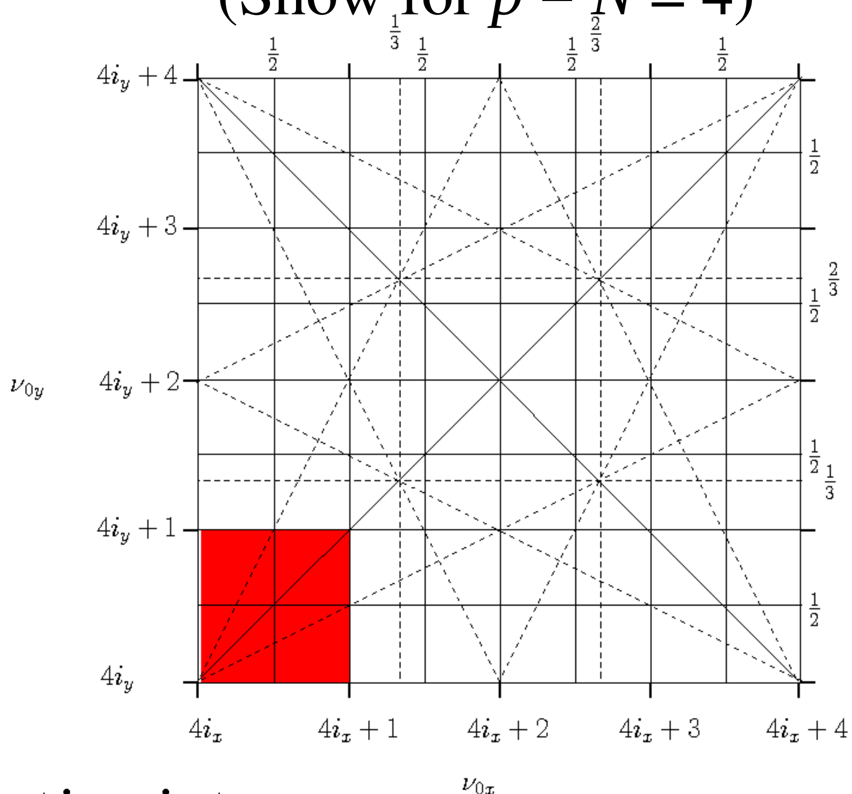
Random Perturbations

(p = 1)



Systematic Perturbations

(Show for p = N = 4)



$$i_x, i_y = \text{postive integers}$$

Discussion:

Random errors:

- Errors always present and give low-order resonances
- Usually have weak amplitude coefficients
  - Can be corrected to reduce effects

Systematic errors:

- Lead to higher-order resonances for large N and a lower density of resonance lines
  - Large symmetric rings with high N values have less operating restrictions from systematic errors
  - Practical issues such as construction cost and getting the beam into and out of the ring can lead to smaller N values
- BUT Amplitude coefficients can be large
  - Systematic effects accumulate in amplitude period by period

Resonances beyond 3rd order rarely need be considered

- Effects outside of model assumed tend to wash-out higher order resonances

More detailed treatments calculate amplitudes/strengths of resonant terms

- See accelerator physics references: ....

S8: Space-Charge and Other Effects Altering Resonances

Ring operating points are generally chosen to be far from low-order resonance lines in x-y tune space. Processes that act to shift resonances closer towards the low-order lines can prove problematic:

- Oscillation amplitudes increase (spoiling beam quality and control)
- Particles can be lost

Tune shift limits of machine operation are often named "Laslett Limits" in honor of Jackson Laslett who first calculated tune shift limits for many processes:

- Image charges
- Image currents
- KV model linear self-fields internal to the beam
- ...

Processes shifting resonances can be grouped into two broad categories:

Coherent — Same for every particle in distribution
- Usually most dangerous

Incoherent — Different for particles in separate parts of the distribution
- Usually less dangerous: only effects part of beam

Laslett space-charge limit

Laslett first obtained a space-charge limit for rings by assuming that the beam space-charge is uniformly distributed as in a KV model. Denote:

$$\nu_{0x} \equiv x\text{-tune (bare) in absence of space-charge}$$

$$\nu_x \equiv x\text{-tune (depressed) with uniform density beam}$$

$$\Delta\nu_x \equiv \nu_{0x} - \nu_x = \text{Space-charge tune shift} \qquad \Delta\nu_x \geq 0$$

Assume that dipole (integer) and quadrupole (half-integer) tunes only need be excluded when space-charge effects are included.

- Space-charge likely induces more washing-out of higher order resonances

Then if the bare tune operating point is chosen as far as possible from ½ -integer resonance lines, the maximum space-charge tune shift allowed is taken to be ¼-integer, giving:

$$\Delta\nu_x|_{\max} = \frac{1}{4} \quad \Longrightarrow$$

Establishes maximum current (use KV results in Transverse Equilibrium Distributions)

- Analogous equation applies in the *y*-plane
  - Identical restriction in lattices with equal *x*- and *y*-focusing strengths

Discussion:

Laslett limit may be overly restrictive:

- KV model assumes all particles in beam have the same tune
  - Significant spectrum of particle tunes likely in real beam
  - No equilibrium beam: core oscillates and space-charge may act incoherently to effectively wash-out resonances
- Simulations suggest Laslett limit poses little issues over 10s – 100s of laps in rings
  - Longer simulations very difficult to resolve: see Simulation Techniques
- Future experiments can hopefully address this issue
  - University of Maryland electron ring will have strong space-charge

More research on this topic is needed!

- Higher intensities can open new applications for energy and material processing
- Many possibilities to extend operating range of existing machines and make new use of developed technology
- Good area for graduate thesis projects!

These notes will be corrected and expanded for reference and future editions of US Particle Accelerator School and University of California at Berkeley courses:

"Beam Physics with Intense Space Charge"

"Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields"

by J.J. Barnard and S.M. Lund

Corrections and suggestions for improvements are welcome. Contact:

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References: For more information see:

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