Transverse Equilibrium Distributions*

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USPAS: "Beam Physics with Intense Space-Charge"
UCB: "Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields"

US Particle Accelerator School (USPAS)
University of California at Berkeley (UCB)

Nuclear Engineering Department NE 290H
Spring Semester, 2009
(Version 20090304)

* Research supported by the US Dept. of Energy at LLNL and LBNL under contract Nos. DE-AC52-07NA27344 and DE-AC02-05CH11231.

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Transverse Equilibrium Dist. Functions: Detailed Outline

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S1: Transverse Vlasov-Poisson Model: for a coasting, single species beam with electrostatic self-fields propagating in a linear focusing lattice:

$\mathbf{x}_\perp, \ \mathbf{x}'_\perp$ transverse particle coordinate, angle

$q, \ m$ charge, mass

$\gamma_b, \ \beta_b$ axial relativistic factors

$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle distribution

$H_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle Hamiltonian

Vlasov Equation (see J.J. Barnard, Introductory Lectures):

$$\frac{d}{ds} f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds} \cdot \frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

Particle Equations of Motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad\qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Hamiltonian (see S.M. Lund, lectures on Transverse Particle Equations of Motion):

$$H_\perp = \frac{1}{2}{\mathbf{x}'_\perp}^2 + \frac{1}{2}\kappa_x(s)x^2 + \frac{1}{2}\kappa_y(s)y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Poisson Equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\epsilon_0}\int d^2\mathbf{x}'_\perp \ f_\perp$$

+ boundary conditions on ϕ

Hamiltonian expression of the Vlasov equation:

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

$$= \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

Using the equations of motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} = \mathbf{x}'_\perp$$

$$\frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp} = -\left(\kappa_x x\hat{\mathbf{x}} + \kappa_y y\hat{\mathbf{y}} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right)$$

$$\boxed{\frac{\partial f_\perp}{\partial s} + \mathbf{x}'_\perp\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \left(\kappa_x x\hat{\mathbf{x}} + \kappa_y y\hat{\mathbf{y}} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\right)\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0}$$

In formal dynamics, a “Poisson Bracket” notation is often employed:

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

$$\equiv \frac{\partial f_\perp}{\partial s} + \underbrace{\{H_\perp, f_\perp\}}_{\text{Poisson Bracket}} = 0$$

Comments on Vlasov-Poisson Model

- Collisionless Vlasov-Poisson model good for intense beams with many particles
  - Collisions negligible, see: J.J. Barnard, Intro. Lectures
- Vlasov-Poisson model can be solved as an initial value problem

$$1)\ f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s = s_i) = \text{Initial "condition" (function) specified}$$

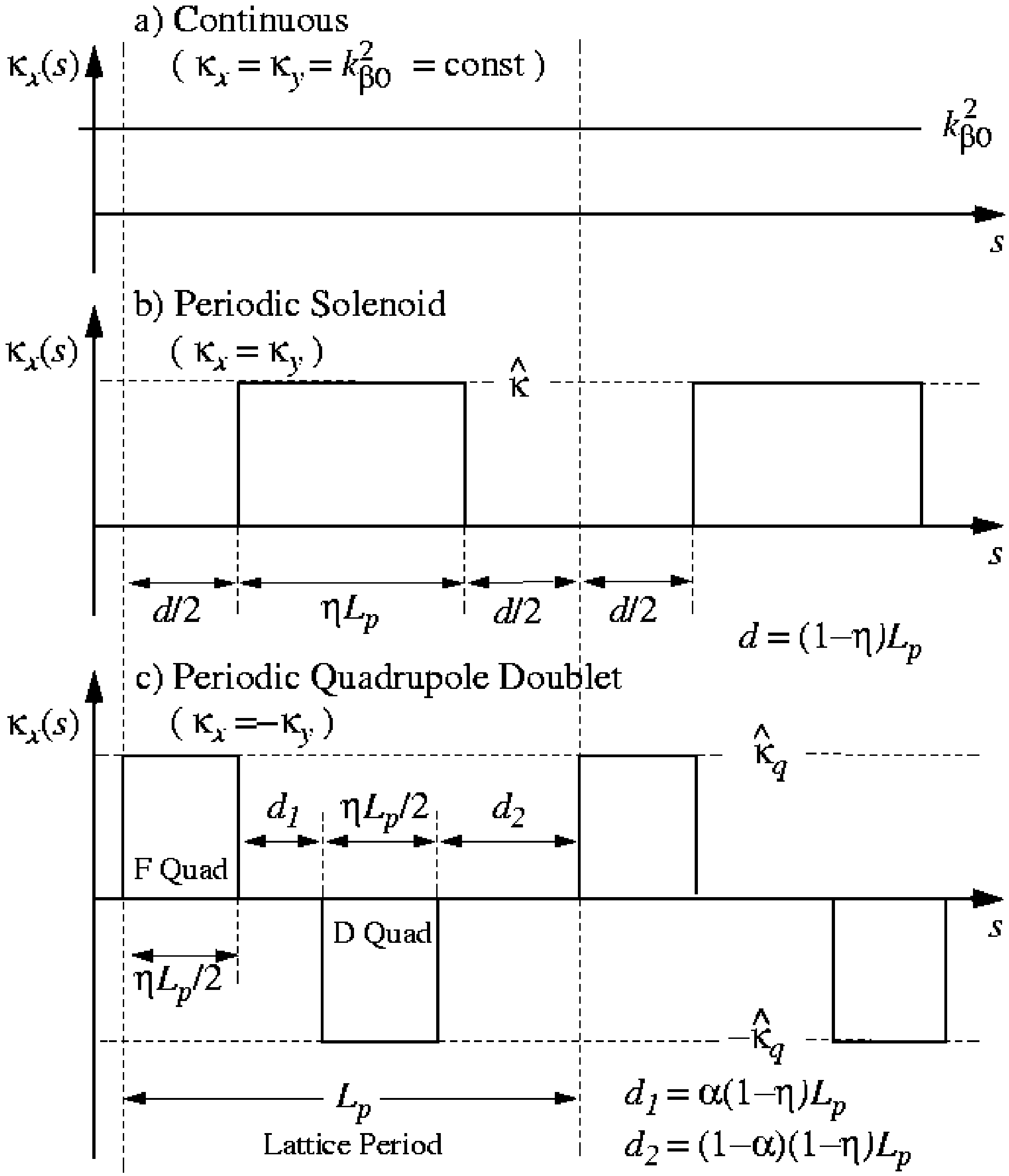
$$2)\ \text{Vlasov-Poisson model solved for subsequent evolution in } s \text{ for } f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) \text{ for } s \geq s_i$$

- The coupling to the self-field via the Poisson equation makes the Vlasov-Poisson model *highly* nonlinear

$$\rho = q \int d^2x'_\perp\ f_\perp \qquad\qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0}$$

- Vlasov-Poisson system is written without acceleration, but the transforms developed to identify the normalized emittance in the lectures on Transverse Particle Equations of Motion can be exploited to generalize all result presented to (weakly) accelerating beams (interpret in tilde variables)
- For solenoidal focusing the system must be interpreted in the rotating Larmor Frame, see: lectures on Transverse Particle Equations of Motion

Review: Focusing lattices, continuous and periodic (simple piecewise constant):



Lattice Period L_p

Occupancy η

$$\eta \in [0, 1]$$

Solenoid description carried out implicitly in Larmor frame
[see: S.M. Lund, lectures on Transverse Particle Equations]

Syncopation Factor α

$$\alpha \in [0, \frac{1}{2}]$$

$$\alpha = \frac{1}{2} \implies FODO$$

Example Hamiltonians:

Continuous focusing: $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Solenoidal focusing: (in Larmor frame variables) $\kappa_x = \kappa_y = \kappa(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}\kappa\mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Quadrupole focusing: $\kappa_x = -\kappa_y = \kappa(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}\kappa x^2 - \frac{1}{2}\kappa y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Review: Undepressed particle phase advance σ_0 is typically employed to characterize the applied focusing strength of periodic lattices:
see: S.M. Lund lectures on Transverse Particle Equations of Motion

x-orbit without space-charge satisfies Hill's equation

$$x''(s) + \kappa_x(s)x(s) = 0$$

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = \mathbf{M}_x(s \mid s_i) \cdot \begin{pmatrix} x(s_i) \\ x'(s_i) \end{pmatrix}$$

$\mathbf{M}_x =$ 2 x 2 Transfer Matrix from $s = s_i$ to s

Undepressed phase advance

$$\boxed{\cos\sigma_{0x} = \frac{1}{2}\mathrm{Tr}\ \mathbf{M}_x(s_i + L_p|s_i)}$$

- Subscript $0x$ used stresses x-plane value and zero ($Q = 0$) space-charge effects

Single particle (and centroid) stability requires:

$$\frac{1}{2}|\mathrm{Tr}\ \mathbf{M}_x(s_i + L_p|s_i)| < 1 \longrightarrow \boxed{\sigma_{0x} < 180^\circ}$$

[Courant and Snyder, Annals of Phys. **3**, 1 (1958)]

Analogous equations hold in the y-plane

The undepressed phase advance can also be equivalently calculated from:

$$w''_{0x} + \kappa_x w_{0x} - \frac{1}{w_{0x}^3} = 0 \qquad w_{0x}(s + L_p) = w_{0x}(s)$$

$$w_{0x} > 0$$

$$\sigma_{0x} = \int_{s_i}^{s_i+L_p} \frac{ds}{w_{0x}^2}$$

- Subscript 0x stresses x-plane value and zero (Q = 0) space-charge effects

S2: Vlasov Equilibria: Plasma physics-like approach is to resolve the system into an equilibrium + perturbation and analyze stability

Equilibrium constructed from single-particle constants of motion C_i

$$\boxed{f_\perp = f_\perp(\{C_i\}) \geq 0 \quad \Longrightarrow \quad \text{equilibrium}}$$

$$\frac{d}{ds} f_\perp(\{C_i\}) = \sum_i \frac{\partial f_\perp}{\partial C_i} \overbrace{\frac{dC_i}{ds}}^{0} = 0$$

Comments:

- Equilibrium is an exact solution to Vlasov's equation that *does not change* in 4D phase-space functional form as *s* advances
  - Equilibrium distribution periodic in lattice period in periodic lattice
  - Projections of the distribution can evolve in *s* in non-continuous lattices
  - Equilibrium is time independent ($\partial/\partial t = 0$) in continuous focusing
- Requirement of positive $f_\perp(\{C_i\})$ follows from single particle species
- Particle conversation constraints are in the presence of (possibly *s*-varying) applied and space-charge forces
  - Highly non-trivial!
  - Only one exact solution known for *s*-varying focusing using Courant-Snyder invariants: the KV distribution to be analyzed in this lecture

/// Example: Continuous focusing $f_\perp = f_\perp(H_\perp)$

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

no explicit *s* dependence

$$\frac{df_\perp}{ds} = \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp}$$

see problem sets for detailed argument

$$= \frac{\partial f_\perp}{\partial H_\perp}\underbrace{\frac{\partial H_\perp}{\partial s}}_{0} + \frac{\partial f_\perp}{\partial H_\perp}\underbrace{\left(\frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\cdot\frac{\partial H_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial H_\perp}{\partial \mathbf{x}'_\perp}\right)}_{0} = 0$$

Showing that $f_\perp = f_\perp(H_\perp)$ exactly satisfies Vlasov's equation for continuous focusing

- Also, for physical solutions must require: $f_\perp(H_\perp) \geq 0$
  - To be appropriate for single species with positive density
- Huge variety of equilibrium function choices $f_\perp(H_\perp)$ can be made to generate many radically different equilibria
  - Infinite variety in function space
- Does *NOT* apply to systems with *s*-varying focusing $\kappa_x \rightarrow k_{\beta 0}^2$
  - Can provide a rough guide if we can approximate:

///

Typical single particle constants of motion:

Transverse Hamiltonian for continuous focusing:

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi = \text{const}$$
$$k^2_{\beta 0} = \text{const}$$

- Not valid for periodic focusing systems!

Angular momentum for systems invariant under azimuthal rotation:

$$P_\theta = xy' - yx' = \text{const}$$

- Subtle point: This form is really a Canonical Angular Momentum and applies to solenoidal magnetic focusing when the variables are expressed in the rotating Larmor frame (i.e., in the "tilde" variables)
  - see: S.M. Lund, lectures on Transverse Particle Equations

Axial kinetic energy for systems with no acceleration:

$$\mathcal{E} = (\gamma_b - 1)mc^2 = \text{const}$$

- Trivial for a coasting beam with $\gamma_b\beta_b = \text{const}$

More on other classes of constraints later ...

Plasma physics approach to beam physics:

Resolve:

$$f(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = f_\perp(\{C_i\}) + \delta f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$$

equilibrium perturbation $f_\perp \gg |\delta f_\perp|$

and carry out equilibrium + stability analysis

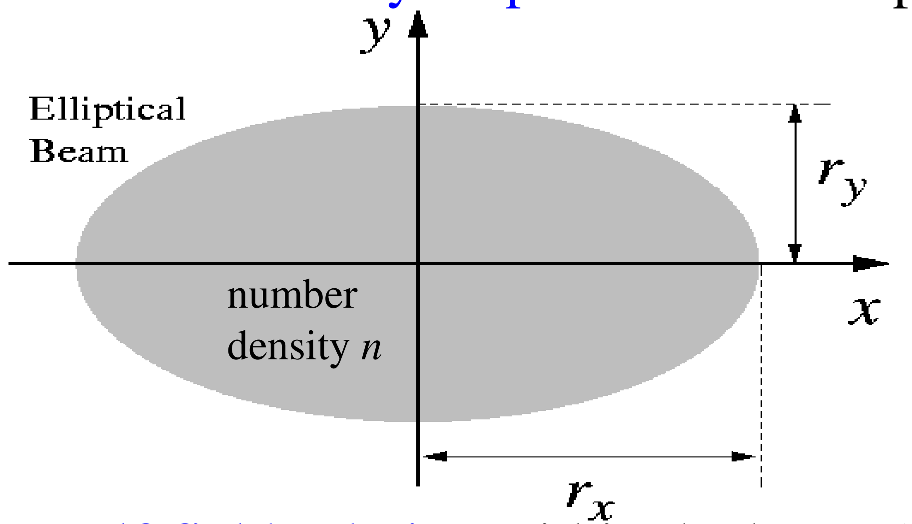
Comments:

- Attraction is to parallel the impressive successes of plasma physics
  - Gain insight into preferred state of nature
- Beams are born off a source and may not be close to an equilibrium condition
  - Appropriate single particle constants of the motion unknown for periodic focusing lattices other than the (unphysical) KV distribution
- Intense beam self-fields and finite radial extent vastly complicate equilibrium description and analysis of perturbations
  - It is not clear if smooth Vlasov equilibria exist (exact sense) in periodic focusing
  - Higher model detail vastly complicates picture!
- If system can be tuned to more closely resemble a relaxed, equilibrium, one might expect less deleterious effects based on plasma physics analogies

S3: The KV Equilibrium Distribution

[Kapchinskij and Vladimirskij, Proc. Int. Conf. On High Energy Accel., p. 274 (1959); and Review: Lund, Kikuchi, and Davidson, PRSTAB, to be published]

Assume a uniform density elliptical beam in a periodic focusing lattice



Line-Charge:

$$\lambda = qn(s)\pi r_x(s) r_y(s)$$
$$= \text{const}$$

Free-space self-field solution within the beam (see: Appendix A) is:

$$\phi = -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{x^2}{(r_x+r_y)r_x} + \frac{y^2}{(r_x+r_y)r_y}\right] + \text{const}$$

$$\boxed{\begin{aligned} -\frac{\partial\phi}{\partial x} &= \frac{\lambda}{\pi\epsilon_0}\frac{x}{(r_x+r_y)r_x} \\ -\frac{\partial\phi}{\partial y} &= \frac{\lambda}{\pi\epsilon_0}\frac{y}{(r_x+r_y)r_y} \end{aligned}}$$

valid only within the beam!

The particle equations of motion:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

become within the beam:

$$\boxed{\begin{aligned} x''(s) + \left\{\kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)}\right\}x(s) = 0 \\ y''(s) + \left\{\kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)}\right\}y(s) = 0 \end{aligned}}$$

Here, Q is the dimensionless perveance defined by:

$$\boxed{Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const}}$$

- Same measure of space-charge intensity used by J.J. Barnard in Intro. Lectures
- Properties/interpretations of the perveance will be extensively developed in in this and subsequent lectures

If we regard the envelope radii r_x, r_y as specified functions of *s*, then these equations of motion are Hill's equations familiar from elementary accelerator physics:

$$x''(s) + \kappa_x^{\text{eff}}(s)x(s) = 0$$

$$y''(s) + \kappa_y^{\text{eff}}(s)y(s) = 0$$

$$\kappa_x^{\text{eff}}(s) = \kappa_x(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_x(s)}$$

$$\kappa_y^{\text{eff}}(s) = \kappa_y(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_y(s)}$$

Suggests Procedure:

- Calculate Courant-Snyder invariants under assumptions made
- Construct a distribution function of Courant-Snyder invariants that generates the uniform density elliptical beam projection assumed
  - Nontrivial step: guess and show that it works

Resulting distribution will be an equilibrium that does not evolve in *s* in 4D phase-space, but lower-dimensional phase-space projections can evolve in *s*

Review (1): The Courant-Snyder invariant of Hill's equation
[Courant and Snyder, Annl. Phys. **3**, 1 (1958)]

Hill's equation describes a zero space-charge particle orbit in linear applied focusing fields:

$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

As a consequence of Floquet's theorem, the solution can be cast in phase-amplitude form:

$$x(s) = A_i w(s) \cos\psi(s)$$

where $w(s)$ is the periodic amplitude function satisfying

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s) \qquad w(s) > 0$$

$\psi(s)$ is a phase function given by

$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

A_i and ψ_i are constants set by initial conditions at $s = s_i$

Review (2): The Courant-Snyder invariant of Hill's equation

From this formulation, it follows that

$$x(s) = A_i w(s) \cos\psi(s)$$

$$x'(s) = A_i w'(s) \cos\psi(s) - \frac{A_i}{w(s)} \sin\psi(s)$$

or

$$\frac{x}{w} = A_i \cos\psi$$

$$wx' - w'x = A_i \sin\psi$$

square and add equations to obtain the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}}$$

- Simplifies interpretation of dynamics
- Extensively used in accelerator physics

Phase-amplitude description of particles evolving within a uniform density beam:

Phase-amplitude form of x-orbit equations:

$$x(s) = A_{xi} w_x(s) \cos\psi_x(s)$$

$$x'(s) = A_{xi} w_x'(s) \cos\psi_x(s) - \frac{A_{xi}}{w_x(s)} \sin\psi_x(s)$$

initial conditions yield:

$$(s = s_i)$$
$$A_{xi} = \text{const}$$
$$\psi_{xi} = \psi_x(s = s_i) = \text{const}$$

where

$$\boxed{\begin{gathered} w_x''(s) + \kappa_x(s) w_x(s) - \frac{2Q}{[r_x(s) + r_y(s)] r_x(s)} w_x(s) - \frac{1}{w_x^3(s)} = 0 \\ w_x(s + L_p) = w_x(s) \qquad w_x(s) > 0 \\ \psi_x(s) = \psi_{xi} + \int_{s_i}^{s} \frac{d\tilde{s}}{w_x^2(\tilde{s})} \end{gathered}}$$

identifies the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_{xi}^2 = \text{const}}$$

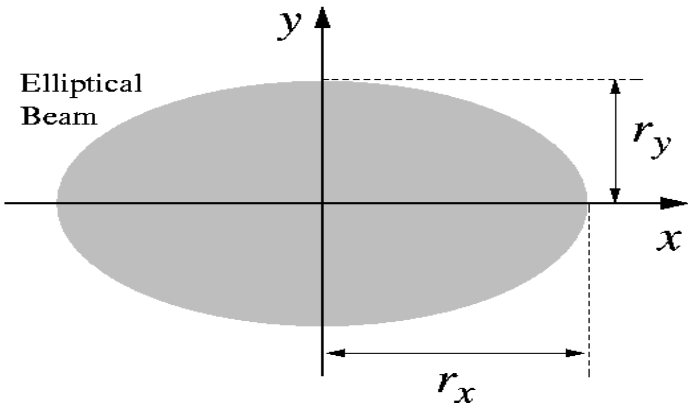
Analogous equations hold for the y-plane

The KV envelope equations:

Define *maximum* Courant-Snyder invariants:

$$\boxed{\begin{aligned}\varepsilon_x &\equiv \mathrm{Max}(A_{xi}^2) \\ \varepsilon_y &\equiv \mathrm{Max}(A_{yi}^2)\end{aligned}}$$

These values must correspond to the beam-edge:



$$r_x(s) = \sqrt{\varepsilon_x}w_x(s)$$
$$r_y(s) = \sqrt{\varepsilon_y}w_y(s)$$

The equations for w_x and w_y can then be rescaled to obtain the familiar KV envelope equations for the matched beam envelope

$$\boxed{\begin{aligned}
&r_x''(s) + \kappa_x(s)r_x(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0 \\
&r_y''(s) + \kappa_y(s)r_y(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0 \\
&r_x(s+L_p) = r_x(s) \qquad\qquad r_x(s) > 0 \\
&r_y(s+L_p) = r_y(s) \qquad\qquad r_y(s) > 0
\end{aligned}}$$

Use variable rescalings to denote *x*- and *y*-plane Courant-Snyder invariants as:

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_{xi}^2 = \text{const}$$

$$\boxed{\begin{aligned}\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 = C_x = \text{const}\\ \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 = C_y = \text{const}\end{aligned}}$$

Kapchinskij and Vladimirskij constructed a delta-function distribution of a linear combination of these Courant-Snyder invariants that generates the correct uniform density elliptical beam needed for consistency with the assumptions:

$$\boxed{f_\perp = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[C_x + C_y - 1\right]}$$

- Delta function means the sum of the x- and y-invariants is a constant
- Other forms cannot generate the needed uniform density elliptical beam projection (see: S9)
- Density inversion theorem covered later can be used to derive result

The KV equilibrium is constructed from the Courant-Snyder invariants:

KV equilibrium distribution:

$$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r'_x x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r'_y y}{\varepsilon_y}\right)^2 - 1\right]$$

$\delta(x) =$ Dirac delta function

This distribution generates (see: proof in Appendix B) the correct uniform density elliptical beam:

$$n = \int d^2x'_\perp \, f_\perp = \begin{cases} \frac{\lambda}{q\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

Obtaining this form consistent with the assumptions, thereby demonstrating full self-consistency of the KV equilibrium distribution.

- Full 4-D form of the distribution does not evolve in *s*
- Projections of the distribution can (and generally do!) evolve in *s*

/// Comment on notation of integrals:

- 2nd forms useful for systems with azimuthal spatial or annular symmetry

Spatial

$$\int d^2 x_\perp \cdots \equiv \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \cdots$$

$$= \int_0^{\infty} dr \; r \int_{-\pi}^{\pi} d\theta \; \cdots$$

Cylindrical Coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Angular

$$\int d^2 x'_\perp \cdots \equiv \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \; \cdots$$

$$= \int_0^{\infty} d\tilde{r}' \; \tilde{r}' \int_{-\pi}^{\pi} d\tilde{\theta}' \; \cdots$$

Angular
Cylindrical Coordinates:

$$x' = \tilde{r}'\cos\tilde{\theta}'$$

$$y' = \tilde{r}'\sin\tilde{\theta}'$$

Use care when interpreting dimensions of symbols in cylindrical form of angular integrals:

$$\tilde{r}' \neq \frac{d}{ds} r = \frac{d}{ds}\sqrt{x^2 + y^2} \qquad [[\tilde{r}']] = \text{Angle} \qquad \tilde{r}' \in [0, \infty)$$

$$\tilde{\theta}' \neq \frac{d}{ds}\theta = \frac{d}{ds}\text{ArcTan}[y, x] \qquad [[\tilde{\theta}']] = \text{rad} \qquad \tilde{\theta}' \in [-\pi, \pi]$$

$$x' = \tilde{r}' \cos\tilde{\theta}' \qquad [[x']] = \text{Angle} \qquad x' \in (-\infty, \infty)$$

$$y' = \tilde{r}' \sin\tilde{\theta}' \qquad [[y']] = \text{Angle} \qquad y' \in (-\infty, \infty)$$

- Tilde is used in angular cylindrical variables to stress that cylindrical variables are chosen in form to span the correct ranges in x' and y' but are not d/ds of the usual cylindrical polar coordinates!

Comment on notation of integrals (continued):

Axisymmetry simplifications

Spatial: for some function $f(\mathbf{x}_\perp^2) = f(r^2)$

$$\int d^2x_\perp \; f(\mathbf{x}_\perp^2) = 2\pi \int_0^\infty dr \; r f(r^2)$$

$$= \pi \int_0^\infty dr^2 \; f(r^2)$$

$$= \pi \int_0^\infty dw \; f(w)$$

Cylindrical Coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$w = r^2$$

Angular: for some function $g(\mathbf{x}_\perp'^2) = g(\tilde{r}'^2)$

$$\int d^2x'_\perp \; g(\mathbf{x}_\perp'^2) = 2\pi \int_0^\infty d\tilde{r}' \; \tilde{r}' g(\tilde{r}'^2)$$

$$= \pi \int_0^\infty d\tilde{r}'^2 \; g(\tilde{r}'^2)$$

$$= \pi \int_0^\infty du \; g(u)$$

Angular
Cylindrical Coordinates:

$$x' = \tilde{r}'\cos\tilde{\theta}'$$

$$y' = \tilde{r}'\sin\tilde{\theta}'$$

$$u = \tilde{r}'^2$$

///

Moments of the KV distribution can be calculated directly from the distribution to further aid interpretation: [see: Appendix B for details]

Full 4D average:

$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \ \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \ f_\perp}$$

Restricted angle average:

$$\langle \cdots \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \ \cdots f_\perp}{\int d^2x'_\perp \ f_\perp}$$

Envelope edge radius:

$$r_x = 2\langle x^2 \rangle_\perp^{1/2}$$

Envelope edge angle:

$$r'_x = 2\langle xx' \rangle_\perp / \langle x^2 \rangle_\perp^{1/2}$$

rms edge emittance (maximum Courant-Snyder invariant):

$$\varepsilon_x = 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp^2]^{1/2} = \text{const}$$

Coherent flows (within the beam, zero otherwise):

$$\langle x' \rangle_{\mathbf{x}'_\perp} = r'_x \frac{x}{r_x}$$

Angular spread (x-temperature, within the beam, zero otherwise):

$$T_x \equiv \langle (x' - \langle x' \rangle_{\mathbf{x}'_\perp})^2 \rangle_{\mathbf{x}'_\perp} = \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)$$

Summary of 1st and 2nd order moments of the KV distribution:

Moment	Value
$\int d^2x'_\perp\ x' f_\perp$	$r'_x \frac{x}{r_x} n$
$\int d^2x'_\perp\ y' f_\perp$	$r'_y \frac{y}{r_y} n$
$\int d^2x'_\perp\ x'^2 f_\perp$	$\left[r'^2_x \frac{x^2}{r_x^2} + \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] n$
$\int d^2x'_\perp\ y'^2 f_\perp$	$\left[r'^2_y \frac{y^2}{r_y^2} + \frac{\varepsilon_y^2}{2r_y^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] n$
$\int d^2x'_\perp\ xx' f_\perp$	$\frac{r'_x}{r_x} x^2 n$
$\int d^2x'_\perp\ yy' f_\perp$	$\frac{r'_y}{r_y} y^2 n$
$\int d^2x'_\perp\ (xy' - yx') f_\perp$	0
$\langle x^2 \rangle_\perp$	$\frac{r_x^2}{4}$
$\langle y^2 \rangle_\perp$	$\frac{r_y^2}{4}$
$\langle x'^2 \rangle_\perp$	$\frac{r'^2_x}{4} + \frac{\varepsilon_x^2}{4r_x^2}$
$\langle y'^2 \rangle_\perp$	$\frac{r'^2_y}{4} + \frac{\varepsilon_y^2}{4r_y^2}$
$\langle xx' \rangle_\perp$	$\frac{r_x r'_x}{4}$
$\langle yy' \rangle_\perp$	$\frac{r_y r'_y}{4}$
$\langle xy' - yx' \rangle_\perp$	0
$16[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]$	ε_x^2
$16[\langle y^2\rangle_\perp \langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2]$	ε_y^2

All 1st and 2nd order moments not listed vanish, i.e.,

$$\int d^2x'_\perp\ xy f_\perp = 0$$

$$\langle xy \rangle_\perp = 0$$

see reviews by:

(limit of results presented)
Lund and Bukh, PRSTAB 024801 (2004), Appendix A

S.M. Lund, T. Kikuchi, and R.C. Davidson, submitted PRSTAB

Canonical transformation illustrates KV distribution structure:
[Davidson, Physics of Nonneutral Plasmas, Addison-Wesley (1990), and Appendix B]

Phase-space transformation:

$$\boxed{\begin{aligned} X &= \frac{\sqrt{\varepsilon_x}}{r_x}x \\ X' &= \frac{r_x x' - r_x' x}{\sqrt{\varepsilon_x}} \end{aligned}}$$

$$dx\ dy = \frac{r_x r_y}{\sqrt{\varepsilon_x \varepsilon_y}} dX\ dY$$

$$dx'\ dy' = \frac{\sqrt{\varepsilon_x \varepsilon_y}}{r_x r_y} dX'\ dY'$$

$$dx\ dy\ dx'\ dy' = dX\ dY\ dX'\ dY'$$

Courant-Snyder invariants in the presence of beam space-charge are then simply:

$$\boxed{X^2 + X'^2 = \text{const}}$$

and the KV distribution takes the simple, *symmetrical* form:

$$\boxed{f_\perp(x, y, x', y', s) = f_\perp(X, Y, X', Y') = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[\frac{X^2 + X'^2}{\varepsilon_x} + \frac{Y^2 + Y'^2}{\varepsilon_y} - 1\right]}$$

from which the density and other projections can be (see: Appendix B) calculated more easily:

$$n = \int d^2x'_\perp\ f_\perp = \frac{\lambda}{q\pi r_x r_y} \int_0^\infty dU^2\ \delta\left[U^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right]$$

$$= \begin{cases} \frac{\lambda}{q\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

KV Envelope equation

The envelope equation reflects low-order force balances

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

Terms:	Applied Focusing	Space-Charge Defocusing	Thermal Defocusing
	Lattice	Perveance	Emittance

Matched Solution:

$$r_x(s + L_p) = r_x(s)$$
$$r_y(s + L_p) = r_y(s)$$

$$\kappa_x(s + L_p) = \kappa_x(s)$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

Comments:

- Envelope equation is a projection of the 4D invariant distribution
  - Envelope evolution equivalently given by moments of the 4D equilibrium distribution
- Most important basic design equation for transport lattices with high space-charge intensity
  - Simplest consistent model incorporating applied focusing, space-charge defocusing, and thermal defocusing forces
  - Starting point of almost all practical machine design!

Comments Continued:

- Beam envelope matching where the beam envelope has the periodicity of the lattice

$$r_x(s + L_p) = r_x(s)$$
$$r_y(s + L_p) = r_y(s)$$

will be covered in much more detail in S.M. Lund lectures on Centroid and Envelope Description of Beams. Envelope matching requires specific choices of initial conditions

$$r_x(s_i),\ r_y(s_i) \qquad\qquad r'_x(s_i),\ r'_y(s_i)$$

for periodic evolution.

- Instabilities of envelope equations are well understood and real (to be covered: see S.M. Lund lectures on Centroid and Envelope Description of Beams)
  - Must be avoided for reliable machine operation

The matched solution to the KV envelope equations reflects the symmetry of the focusing lattice and must in general be calculated numerically

Matching Condition

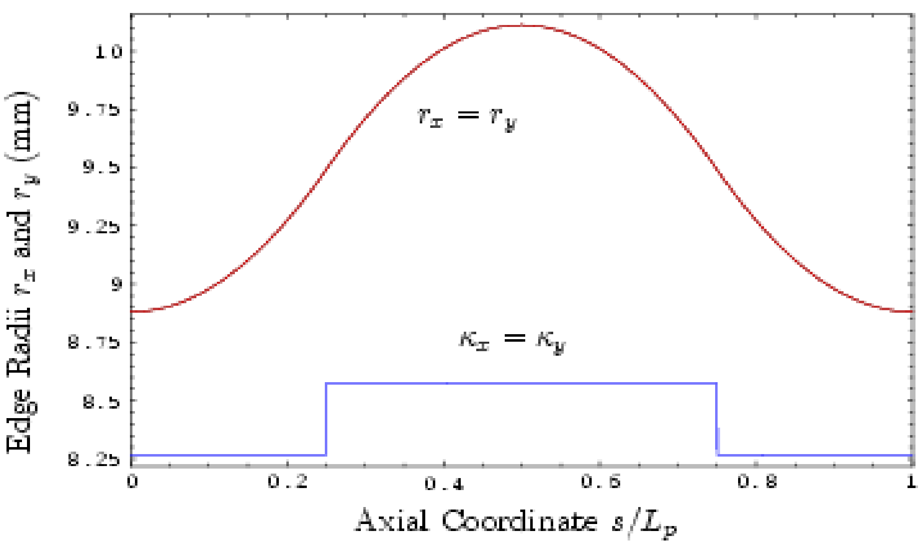
$$\boxed{\begin{aligned} r_x(s + L_p) &= r_x(s) \\ r_y(s + L_p) &= r_y(s) \end{aligned}}$$

Example Parameters

$$L_p = 0.5 \text{ m}, \quad \sigma_0 = 80^\circ, \quad \eta = 0.5$$
$$\varepsilon_x = \varepsilon_y = 50 \text{ mm-mrad}$$
$$\sigma/\sigma_0 = 0.2$$

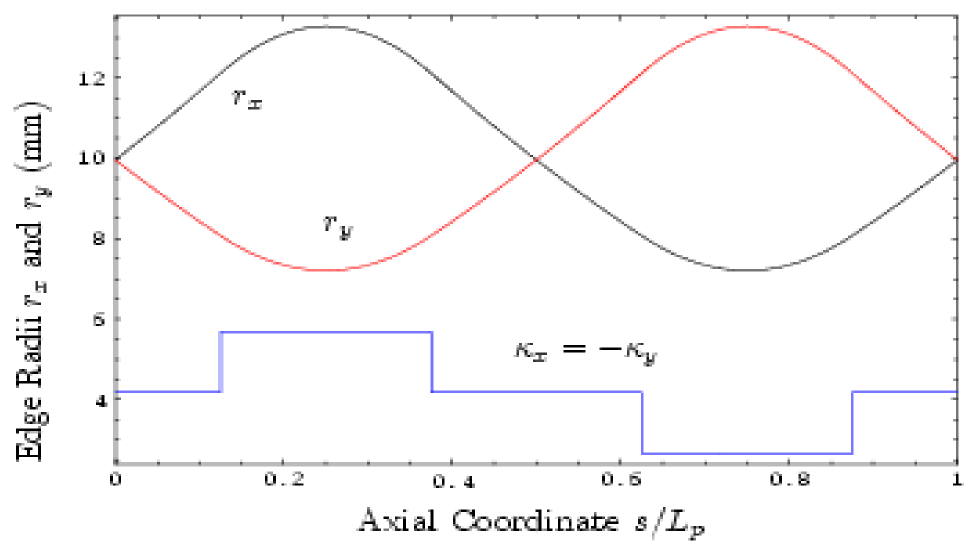
Solenoidal Focusing

$(Q = 6.6986 \times 10^{-4})$



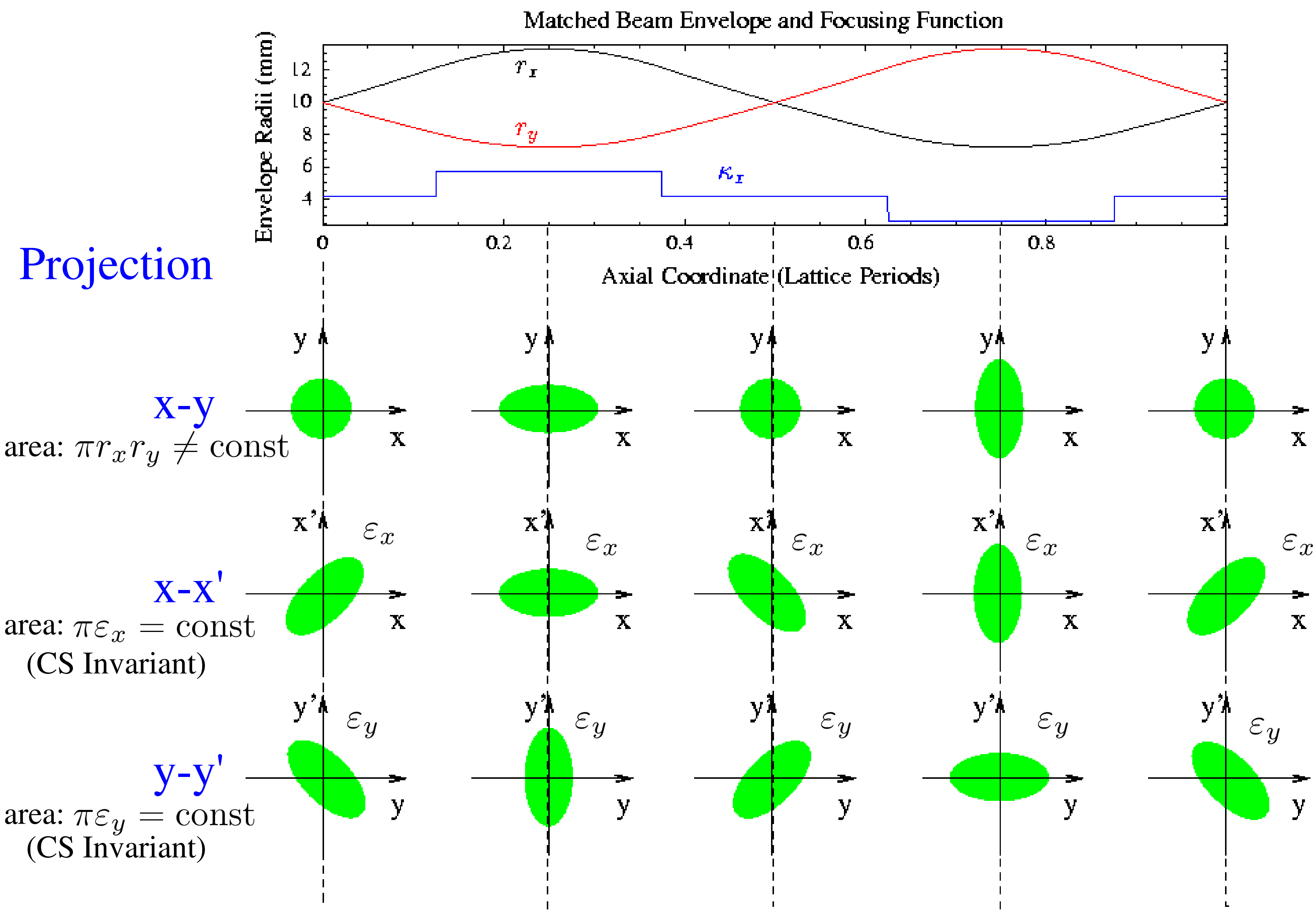
FODO Quadrupole Focusing

$(Q = 6.5614 \times 10^{-4})$



The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

Some phase-space projections of a matched KV equilibrium beam in a periodic FODO quadrupole transport lattice



KV model shows that particle orbits in the presence of space-charge can be strongly modified – space charge slows the orbit response:

Matched envelope:

$$\boxed{\begin{aligned} & r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0 \\ & r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s)+r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0 \\ & r_x(s+L_p) = r_x(s) \qquad r_x(s) > 0 \\ & r_y(s+L_p) = r_y(s) \qquad r_y(s) > 0 \end{aligned}}$$

Equation of motion for x-plane "depressed" orbit in the presence of space-charge:

$$\boxed{x''(s) + \kappa_x(s) x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)} x(s) = 0}$$

All particles have the *same value* of depressed phase advance (similar Eqns in *y*):

$$\boxed{\sigma_x \equiv \psi_x(s_i + L_p) - \psi_x(s_i) = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)}}$$

Contrast: Review, the undepressed particle phase advance calculated in the lectures on Transverse Particle Equations of Motion

The undepressed phase advance in defined as the phase advance of a particle in the absence of space-charge ($Q = 0$):

- Denote by σ_{0x} to distinguished from the "depressed" phase advance σ_x in the presence of space-charge

$$w''_{0x} + \kappa_x w_{0x} - \frac{1}{w^3_{0x}} = 0 \qquad w_{0x}(s + L_p) = w_{0x}(s)$$

$$w_{0x} > 0$$

$$\sigma_{0x} = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2_{0x}}$$

This can be equivalently calculated from the matched envelope with $Q = 0$:

$$r''_{0x} + \kappa_x r_{0x} - \frac{\varepsilon^2_x}{r^3_{0x}} = 0 \qquad r_{0x}(s + L_p) = r_{0x}(s)$$

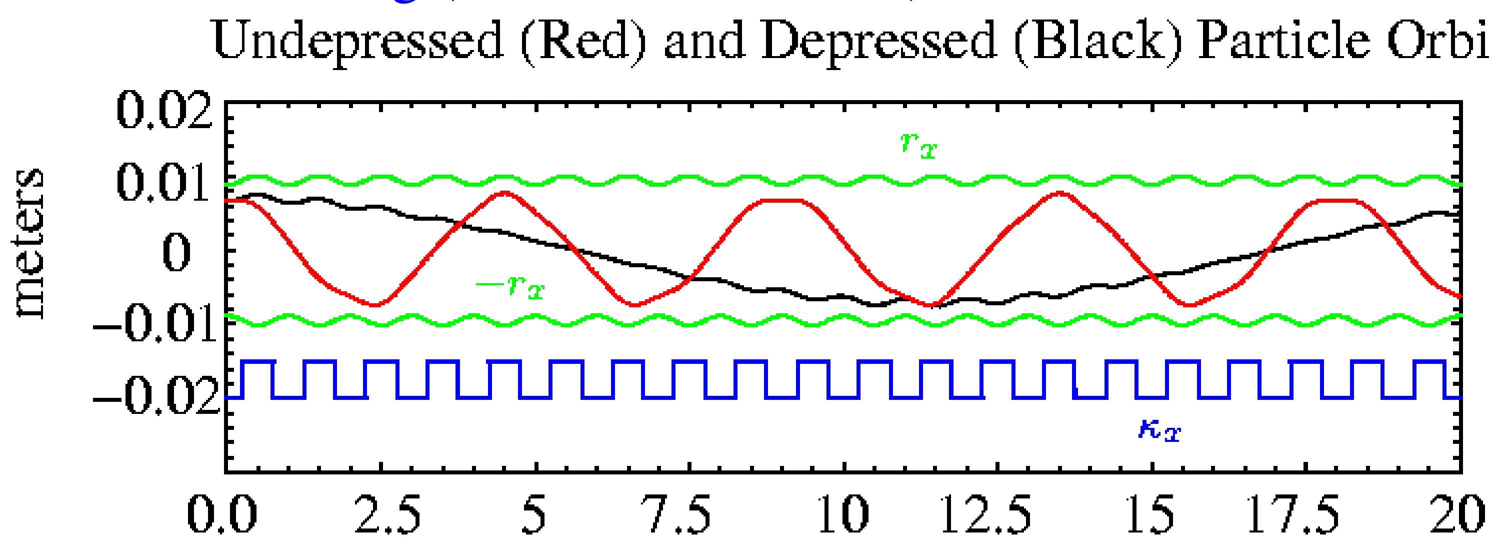
$$r_{0x} > 0$$

$$\sigma_{0x} = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r^2_{0x}}$$

- Value of ε_x is arbitrary (answer for σ_{0x} is independent)

Depressed particle x-plane orbits within a matched KV beam in a periodic FODO quadrupole channel for the matched beams previously shown

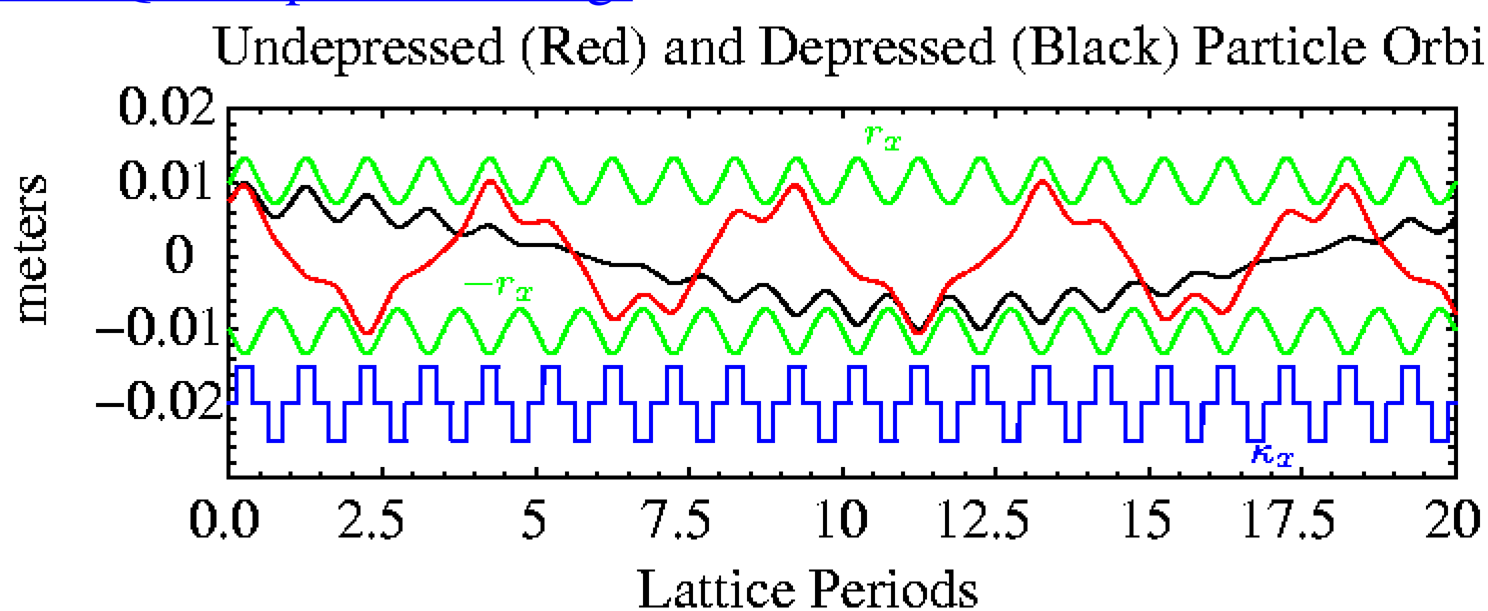
Solenoidal Focusing (Larmor frame orbit):



x-plane orbit:

$y = 0 = y'$

FODO Quadrupole Focusing:



x-plane orbit:

$y = 0 = y'$

Depressed particle phase advance provides a convenient measure of space-charge strength

For simplicity take (plane symmetry in average focusing and emittance)

$$\sigma_{0x} = \sigma_{0y} \equiv \sigma_0 \qquad \qquad \varepsilon_x = \varepsilon_y \equiv \varepsilon$$

Depressed phase advance of particles moving within a matched beam envelope:

$$\boxed{\sigma = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)} = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_y^2(s)}}$$

$$\lim_{Q \to 0} \sigma = \sigma_0$$

Normalized space charge strength

$$\boxed{0 \leq \sigma/\sigma_0 \leq 1}$$

$$\sigma/\sigma_0 \to 0$$

Cold Beam
(space-charge dominated)
$\varepsilon \to 0$

$$\sigma/\sigma_0 \to 1$$

Warm Beam
(kinetic dominated)
$Q \to 0$

For example matched envelope presented earlier:

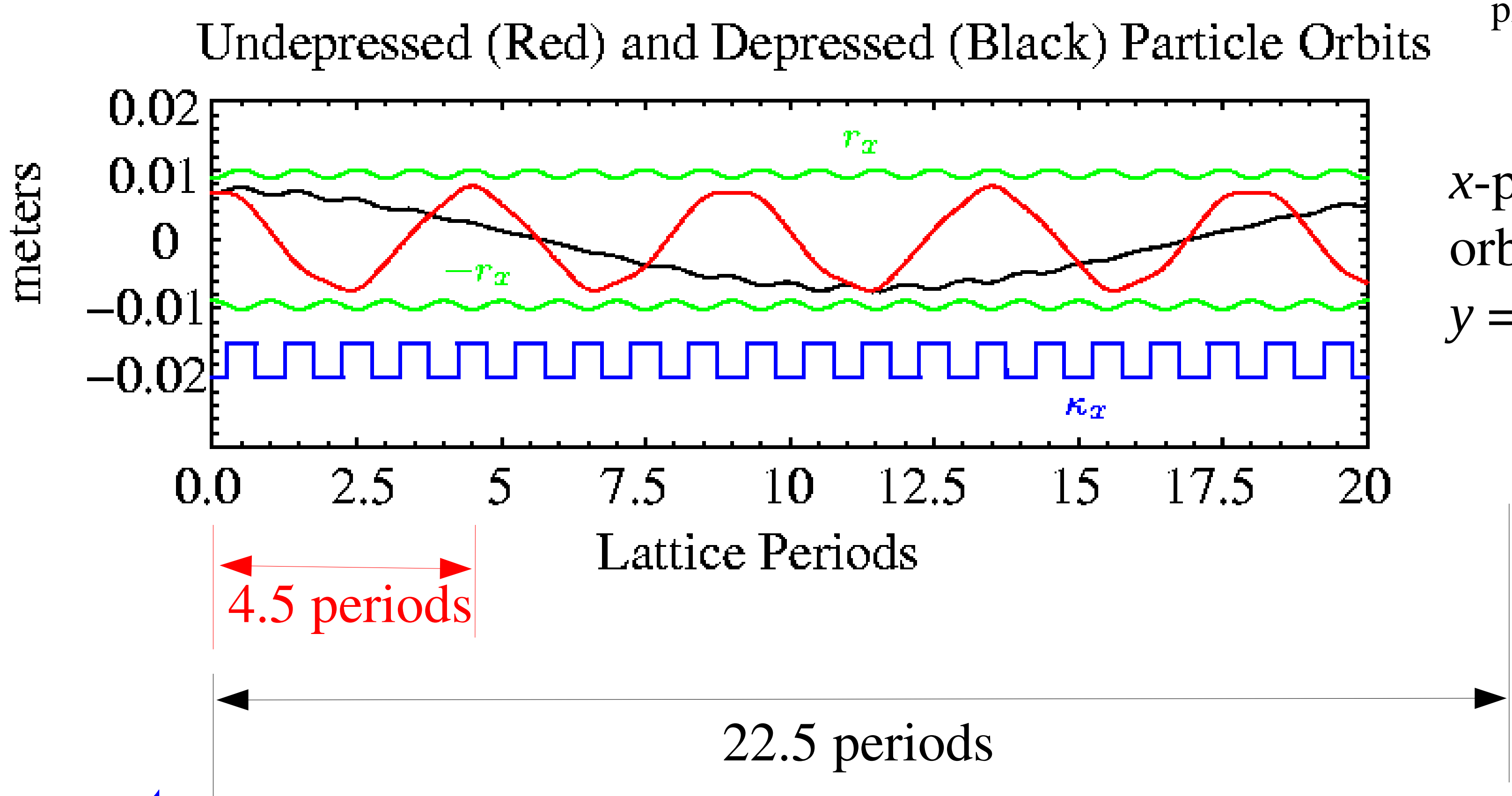
Undepressed phase advance: $\sigma_0 = 80^\circ$

Depressed phase advance: $\sigma = 16^\circ \quad \rightarrow \sigma/\sigma_0 = 0.2$

repeat periods
4.5
22.5

Periods for 360 degree phase advance

Solenoidal Focusing (Larmor frame orbit):



x-plane orbit
$y = 0 = y'$

Comment:

All particles in the distribution will, of course, always move in response to both applied and self-fields. You cannot turn off space-charge for an undepressed orbit. It is a convenient conceptual construction to help understand focusing properties.

The rms equivalent beam model helps interpret general beam evolution in terms of an "equivalent" local KV distribution

Real beams distributions in the lab will not be KV form. But the KV model can be applied to interpret arbitrary distributions via the concept of *rms equivalence*.

For the same focusing lattice, replace any beam charge $\rho(x,y)$ density by a uniform density KV beam of the same species (q, m) and energy (β_b) in each axial slice (s) using averages calculated from the actual "real" beam distribution with:

$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \ \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \ f_\perp} \qquad f_\perp = \text{ real distribution}$$

rms equivalent beam (identical 1st and 2nd order moments):

Quantity	KV Equiv.	Calculated from Distribution
Perveance	Q	$= q^2 \int d^2x_\perp \int d^2x'_\perp \ f_\perp \ /[2\pi\epsilon_0\gamma_b^3\beta_b^2c^2]$
x-Env Rad	r_x	$= 2\langle x^2\rangle_\perp^{1/2}$
y-Env Rad	r_y	$= 2\langle y^2\rangle_\perp^{1/2}$
x-Env Angle	r'_x	$= 2\langle xx'\rangle_\perp/\langle x^2\rangle_\perp^{1/2}$
y-Env Angle	r'_y	$= 2\langle yy'\rangle_\perp/\langle y^2\rangle_\perp^{1/2}$
x-Emittance	ε_x	$= 4[\langle x^2\rangle_\perp\langle x'^2\rangle_\perp - \langle xx'\rangle_\perp]^{1/2}$
y-Emittance	ε_y	$= 4[\langle y^2\rangle_\perp\langle y'^2\rangle_\perp - \langle yy'\rangle_\perp]^{1/2}$

Comments on rms equivalent beam concept:

- The emittances will generally evolve in s
  - Means that the equivalence must be recalculated in every slice as the emittances evolve
  - For reasons to be analyzed later (see S.M. Lund lectures on Kinetic Stability of Beams), this evolution is often small
- Concept is highly useful
  - KV equilibrium properties well understood and are approximately correct to model lowest order "real" beam properties
  - See, Reiser, *Theory and Design of Charged Particle Beams* (1994, 2008) for a detailed discussion of rms equivalence

Sacherer expanded the concept of rms equivalency by showing that the equivalency works exactly for beams with elliptic symmetry space-charge [Sacherer, IEEE Trans. Nucl. Sci. 18, 1101 (1971), J.J. Barnard, Intro. Lectures]

For any beam with elliptic symmetry charge density in each transverse slice:

$$\boxed{\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)}$$

Based on:

$$\langle x \frac{\partial \phi}{\partial x} \rangle_\perp = -\frac{\lambda}{4\pi\epsilon_0} \frac{r_x}{r_x + r_y}$$

see J.J. Barnard intro. lectures

the KV envelope equations

$$\boxed{\begin{aligned} r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2(s)}{r_x^3(s)} &= 0 \\ r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2(s)}{r_y^3(s)} &= 0 \end{aligned}}$$

remain valid when (averages taken with the full distribution):

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const} \qquad \lambda = q \int d^2 x_\perp \, \rho \; = \; \text{const}$$

$$r_x = 2\langle x^2 \rangle_\perp^{1/2} \qquad \varepsilon_x = 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle x x' \rangle_\perp^2]^{1/2}$$

$$r_y = 2\langle y^2 \rangle_\perp^{1/2} \qquad \varepsilon_y = 4[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle y y' \rangle_\perp^2]^{1/2}$$

The emittances must, in general, evolve in *s* under this model

(see SM Lund lectures on *Transverse Kinetic Stability*)

Interpretation of the dimensionless perveance Q

The dimensionless perveance:

$$\boxed{Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const}}$$

$$\lambda = q\hat{n}\pi r_x r_y = \text{line-charge} = \text{const}$$

$$\hat{n} = \text{beam density}$$

- Scales with size of beam (λ), but typically has small characteristic values even for beams with high space charge intensity (~ 10^{-4} to 10^{-8} common)
- Even small values of Q can matter depending on the relative strength of other effects from applied focusing forces, thermal defocusing, etc.

Can be expressed equivalently in several ways:

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \frac{qI_b}{2\pi\epsilon_0 m\gamma_b^3\beta_b^3c^3} = \frac{1}{(\gamma_b\beta_b)^3}\frac{I_b}{I_A}$$

$$= \frac{q^2\pi r_x r_y\hat{n}}{2\pi\epsilon_0 m\gamma_b^3\beta_b^3c^3} = \frac{\hat{\omega}_p^2 r_x r_y}{2\gamma_b^3\beta_b^2c^2}$$

$$I_b = \lambda\beta_b c = \text{beam current}$$

$$I_A = 4\pi\epsilon_0 mc^3/q = \text{Alfven current}$$

$$\hat{\omega}_p = \sqrt{q^2\hat{n}/(m\epsilon_0)} = \text{plasma freq.}$$

- Forms based on $\lambda,\ I_b$ generalize to *nonuniform density beams*

To better understand the perveance Q, consider a round, uniform density beam with

$$r_x = r_y = r_b$$

then the solution for the potential within the beam reduces:

$$\begin{aligned}\phi &= -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{x^2}{(r_x+r_y)r_x} + \frac{y^2}{(r_x+r_y)r_y}\right] + \text{const} \\ &= -\frac{\lambda}{4\pi\epsilon_0}\frac{r^2}{r_b^2} + \text{const}\end{aligned}$$

$$\Longrightarrow \quad \Delta\phi = \phi(r=0) - \phi(r=r_b) = \frac{\lambda}{4\pi\epsilon_0}$$

for potential drop across the beam

If the beam is also nonrelativistic, then the axial kinetic energy $\mathcal{E}_b$ is

$$\mathcal{E}_b = (\gamma_b - 1)mc^2 \simeq \frac{1}{2}m\beta_b^2 c^2$$

and the perveance can be alternatively expressed as

$$\boxed{Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} \simeq \frac{q\Delta\phi}{\mathcal{E}_b}}$$

- Perveance can be interpreted as space-charge potential energy difference across beam relative to the axial kinetic energy

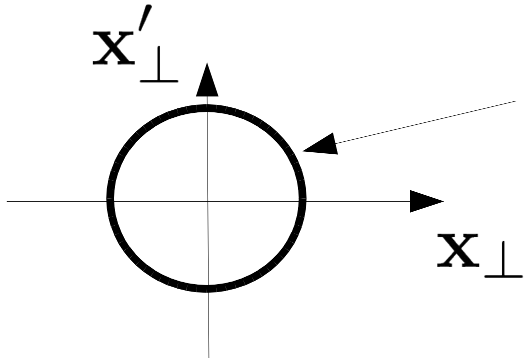
Further comments on the KV equilibrium: Distribution Structure

KV equilibrium distribution:

$$\boxed{f_\perp \sim \delta[\text{Courant-Snyder invariants}]}$$

Forms a highly singular hyper-shell in 4D phase-space

Schematic:



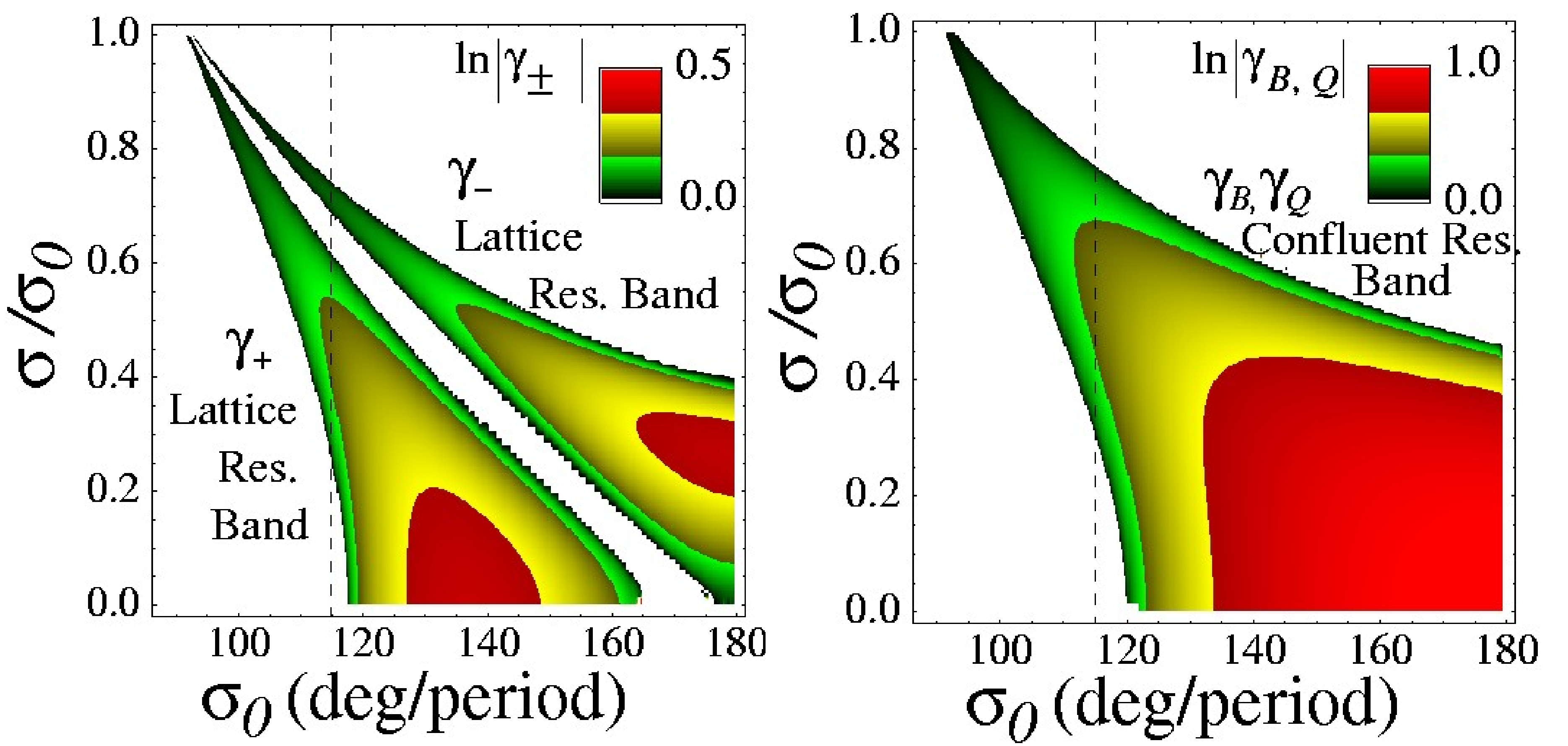
- Singular distribution has large “Free-Energy” to drive many instabilities
  - Low order envelope modes are physical and highly important (see: lectures by S.M. Lund on Centroid and Envelope Descriptions of Beams)
- Perturbative analysis shows strong collective instabilities
  - Hofmann, Laslett, Smith, and Haber, Part. Accel. **13**, 145 (1983)
  - Higher order instabilities (collective modes) have unphysical aspects due to (delta-function) structure of distribution and must be applied with care (see: lectures by S.M. Lund on Kinetic Stability of Beams)
  - Instabilities can cause problems if the KV distribution is employed as an initial beam state in self-consistent simulations

Preview: lecture on Centroid and Envelope Descriptions of Beams:

Instability bands of the KV envelope equation are well understood in periodic focusing channels and must be avoided in machine operation

Envelope Mode Instability Growth Rates

Solenoid (η = 0.25)   Quadrupole FODO (η = 0.70)

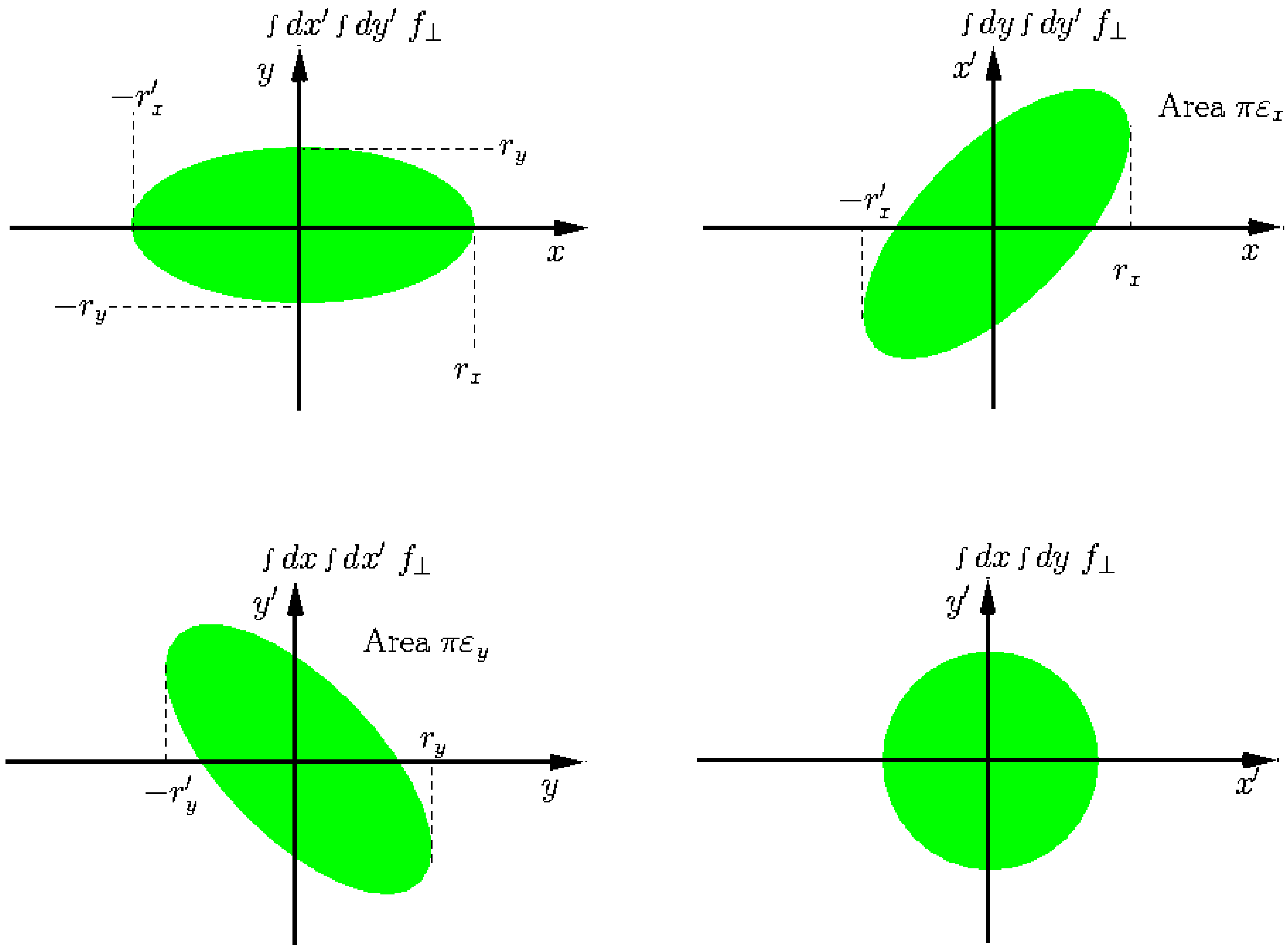


[S.M. Lund and B. Bukh, PRSTAB **7** 024801 (2004)]

Further comments on the KV equilibrium: 2D Projections

All 2D projections of the KV distribution are uniformly filled ellipses

- Not very different from what is often observed in experimental measurements and self-consistent simulations of stable beams with strong space-charge
- Falloff of distribution at "edges" can be rapid, but smooth, for strong space-charge



Further comments on the KV equilibrium:
Angular Spreads: Coherent and Incoherent

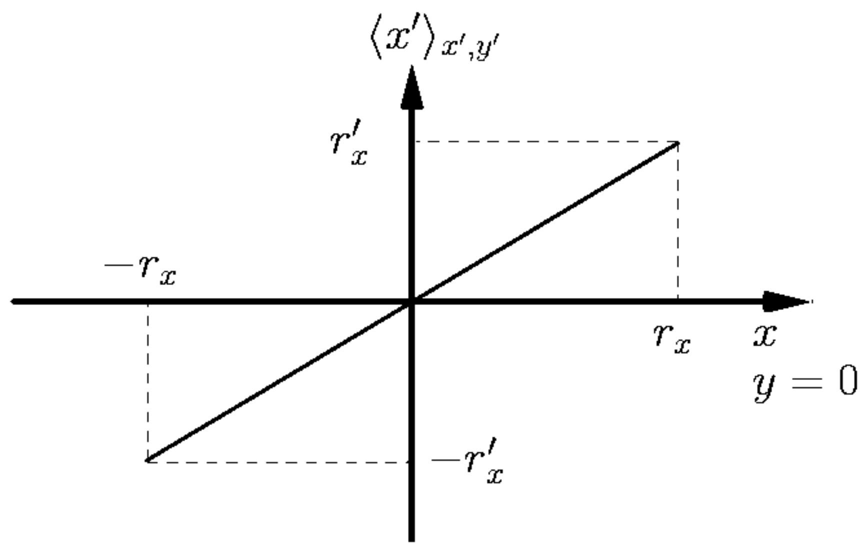
Angular spreads within the beam:

Coherent (flow):

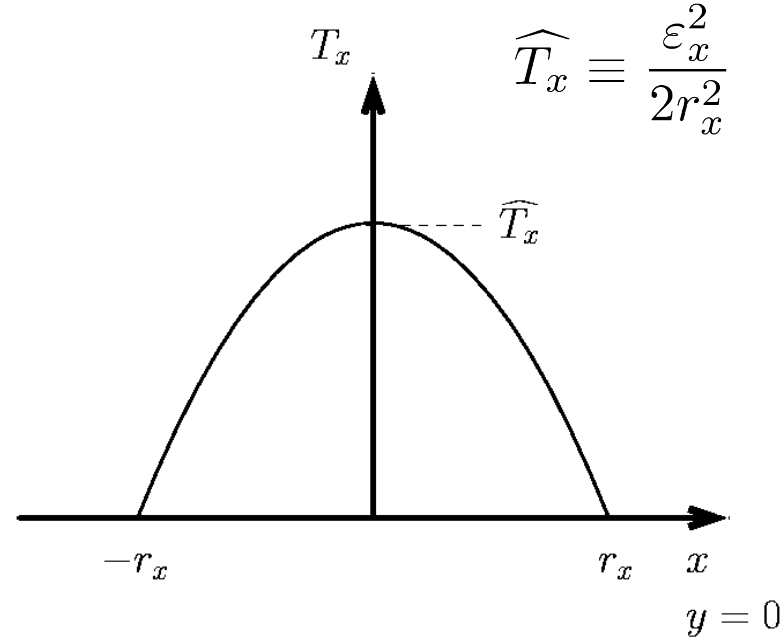
$$\langle x' \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \; x'_\perp \; f_\perp}{\int d^2x'_\perp \; f_\perp} = r'_x \frac{x}{r_x}$$

Incoherent (temperature):

$$\langle (x' - r'_x x/r_x)^2 \rangle_{\mathbf{x}'_\perp} = \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)$$



$$\widehat{T_x} \equiv \frac{\varepsilon_x^2}{2r_x^2}$$



- Coherent flow required for periodic focusing to conserve charge
- Temperature must be zero at the beam edge since the distribution edge is sharp
- Parabolic temperature profile is consistent with linear grad P pressure forces in a fluid model interpretation of the (kinetic) KV distribution

Further comments on the KV equilibrium:

The KV distribution is the *only* exact equilibrium distribution formed from Courant-Snyder invariants of linear forces valid for periodic focusing channels:

- Low order properties of the distribution are physically appealing
- Illustrates relevant Courant-Snyder invariants in simple form
  - Later arguments demonstrate that these invariants should be a reasonable approximation for beams with strong space charge
- KV distribution does not have a 3D generalization [see F. Sacherer, Ph.d. thesis, 1968]

Strong Vlasov instabilities associated with the KV model render the distribution inappropriate for use in evaluating machines at high levels of detail:

- Instabilities are not all physical and render interpretation of results difficult
  - Difficult to separate physical from nonphysical effects in simulations

Possible Research Problem (unsolved in 40+ years!):
Can a valid Vlasov equilibrium be constructed for a *smooth* (non-singular), nonuniform density distribution in a linear, periodic focusing channel?

- Not clear what invariants can be used or if any can exist
  - Nonexistence proof would also be significant
- Lack of a smooth equilibrium does not imply that real machines cannot work!

Because of a lack of theory for a smooth, self-consistent distribution that would be more physically appealing than the KV distribution we will examine smooth distributions in the idealized continuous focusing limit (after an analysis of the continuous limit of the KV theory):

- Allows more classic "plasma physics" like analysis
- Illuminates physics of intense space charge
- Lack of continuous focusing in the laboratory will prevent over generalization of results obtained

A 1D analog to the KV distribution called the "Neuffer Distribution" is useful in longitudinal physics

- Based on linear forces with a "g-factor" model
- Distribution is not singular in 1D
- See: J.J. Barnard, lectures on Longitudinal Physics

Appendix A: Self-Fields of a Uniform Density Elliptical Beam in Free-Space

1) Direct Proof:

The solution to the 2D Poisson equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = \begin{cases} -\frac{\lambda}{\pi\epsilon_0 r_x r_y}, & \text{if } \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1 \\ 0, & \text{if } \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1 \end{cases}$$

$$\lim_{r\to\infty} \frac{\partial\phi}{\partial r} \sim \frac{\lambda}{2\pi\epsilon_0 r}$$

has been formally constructed as:

- Solutions date from early Newtonian gravitational field solutions of stars with ellipsoidal density
- See Landau and Lifshitz, *Classical Theory of Fields* for a simple presentation

$$\phi = -\frac{\lambda}{4\pi\epsilon_0}\left\{\int_0^{\xi} \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}} + \int_{\xi}^{\infty} \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{x^2}{r_x^2+s} + \frac{y^2}{r_y^2+s}\right)\right\} + \text{const}$$

$$\xi = 0 \;\; \text{when} \;\; x^2/r_x^2 + y^2/r_y^2 < 1$$

$$\xi \text{ root of: } \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} = 1, \quad \text{when} \;\; \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} > 1$$

A1

We will A) demonstrate that this solution works and then B) simplify the result.

A) Verify by direct substitution:

$$\frac{\partial \phi}{\partial x} = -\frac{\lambda}{4\pi\epsilon_0}\left\{\int_\xi^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{2x}{r_x^2+s}\right) - \frac{1}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left[1-\frac{x^2}{r_x^2+\xi}-\frac{y^2}{r_y^2+\xi}\right]\frac{\partial \xi}{\partial x}\right\}$$

But:

$$\text{if } \xi = 0 \implies 1 = \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi}$$

$$\text{if } \xi = 0 \implies \frac{d\xi}{dx} = 0$$

$\implies$ In either case the 2nd term above vanishes

Giving:

$$\frac{\partial \phi}{\partial x} = -\frac{\lambda}{2\pi\epsilon_0}\int_\xi^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{x}{r_x^2+s}\right)$$

$$\frac{\partial \phi}{\partial y} = -\frac{\lambda}{2\pi\epsilon_0}\int_\xi^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{y}{r_y^2+s}\right)$$

Differentiate again and apply the chain rule:

A2

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\epsilon_0}\left\{\int_\xi^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left(\frac{1}{r_x^2+s}+\frac{1}{r_y^2+s}\right)\right.$$

$$\left. - \frac{1}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left[\frac{x\partial\xi/\partial x}{r_x^2+\xi}+\frac{y\partial\xi/\partial y}{r_y^2+\xi}\right]\right\}$$

Must show that the right hand side reduces to the required elliptical form for a uniform density beam for:

Case 1: Exterior $\quad \dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi}=1$

Case 2: Interior $\quad \xi = 1$

Case 1: Exterior $\quad \dfrac{x^2}{r_x^2}+\dfrac{y^2}{r_y^2}>1$

Differentiate: $\quad \dfrac{x^2}{r_x^2+\xi}+\dfrac{y^2}{r_y^2+\xi}=1$

$$\Longrightarrow \quad \frac{\partial\xi}{\partial x} = \frac{2x}{(r_x^2+\xi)}\frac{1}{\left[\frac{x^2}{(r_x^2+\xi)^2}+\frac{y^2}{(r_y^2+\xi)^2}\right]}$$

+ analogous eqn in *y*

A3

Using these results:

$$\frac{x\partial\xi/\partial x}{r_x^2+\xi} + \frac{y\partial\xi/\partial y}{r_y^2+\xi} = 2\left[\frac{x^2}{(r_x^2+\xi)^2} + \frac{y^2}{(r_y^2+\xi)^2}\right]\frac{1}{\left[\frac{x^2}{(r_x^2+\xi)^2} + \frac{y^2}{(r_y^2+\xi)^2}\right]} = 2$$

Also, need to calculate integrals like:

$$w^2 = r_x^2 + \tilde{\xi}$$

$$I_x(\xi) \equiv \int_\xi^\infty \frac{d\tilde{\xi}}{[(r_x^2+\tilde{\xi})(r_y^2+\tilde{\xi})]^{1/2}}\frac{1}{r_x^2+\tilde{\xi}} = \int_{\sqrt{r_x^2+\xi}}^\infty \frac{dw}{(r_x^2-r_y^2+w^2)^{3/2}}$$

+ analogous integrals in *y*

This integral can be done using tables or symbolic programs like Mathematica:

$$I_x(\xi) = \left.\frac{2w}{(r_x^2-r_y^2)\sqrt{r_x^2-r_y^2+w^2}}\right|_{w=\sqrt{r_x^2+\xi}}^{w\to\infty} = \frac{2}{r_x^2-r_y^2} + \frac{2\sqrt{r_y^2+\xi}}{(r_x^2-r_y^2)\sqrt{r_x^2+\xi}}$$

Applying this integral and the analogous $I_y(\xi)$

$$\int_0^\infty \frac{ds}{\sqrt{(r_x^2+s)(r_y^2+s)}}\left[\frac{1}{r_x^2+s} + \frac{1}{r_y^2+s}\right] = I_x(\xi) + I_y(\xi)$$

$$= \frac{2}{r_x^2-r_y^2}\left(\frac{\sqrt{r_x^2+\xi}}{\sqrt{r_y^2+\xi}} - \frac{\sqrt{r_y^2+\xi}}{\sqrt{r_x^2+\xi}}\right) = \frac{2}{\sqrt{(r_x^2+\xi)(r_y^2+\xi)}}$$ A4

Applying both of these results, we obtain:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\epsilon_0}\left\{\frac{2}{\sqrt{(r_x^2+\xi)(r_y^2+\xi)}} - \frac{2}{\sqrt{(r_x^2+\xi)(r_y^2+\xi)}}\right\}$$

$$= 0$$ Thereby verifying the exterior case !

Case 2: Interior $\quad \dfrac{x^2}{r_x^2} + \dfrac{y^2}{r_y^2} < 1$

$$\xi = 0 \quad \Longrightarrow \quad \frac{x\partial\xi/\partial x}{r_x^2+\xi} + \frac{y\partial\xi/\partial y}{r_y^2+\xi} = 0$$

The integrals defined and calculated above give in this case:

$$I_x(\xi = 0) = \frac{2}{(r_x + r_y)r_x} \qquad I_y(\xi = 0) = \frac{2}{(r_x + r_y)r_y}$$

Applying both of these results, we obtain:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\lambda}{2\pi\epsilon_0}\left\{\frac{2}{r_x r_y} - 0\right\} = -\frac{\lambda}{\epsilon_0 \pi r_x r_y} = -\frac{q\hat{n}}{\epsilon_0}$$

Thereby verifying the interior case !

A5

Verify that the correct large-r limit of the potential is obtained outside the beam:

$$-\frac{\partial \phi}{\partial x} = \frac{\lambda}{2\pi\epsilon_0} x I_x(\xi)$$
$$-\frac{\partial \phi}{\partial y} = \frac{\lambda}{2\pi\epsilon_0} y I_y(\xi)$$

$$r \text{ large} \Longrightarrow \xi \text{ large}$$

$$\lim_{r\to\infty} I_x(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$
$$\lim_{r\to\infty} I_y(\xi) = \frac{1}{\xi} = \frac{1}{r^2}$$

Thus:

$$\lim_{r\to\infty} -\frac{\partial \phi}{\partial x} = -\frac{\lambda}{2\pi\epsilon_0}\frac{x}{r^2}$$
$$\lim_{r\to\infty} -\frac{\partial \phi}{\partial y} = -\frac{\lambda}{2\pi\epsilon_0}\frac{y}{r^2}$$

$$\Longrightarrow \quad \lim_{r\to\infty} -\frac{\partial \phi}{\partial r} = \frac{\lambda}{2\pi\epsilon_0 r}$$

Thereby verifying the exterior limit!

Together, these results fully verify that the integral solution satisfies the Poisson equation describing a uniform density elliptical beam in free space

A6

Finally, it is useful to apply the steps in the verification to derive a simplified formula for the potential within the beam where:

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1, \quad \xi = 0$$

This gives:

$$\begin{aligned} \phi &= -\frac{\lambda}{4\pi\epsilon_0} \left\{ x^2 I_x(\xi = 0) + y^2 I_y(\xi = 0) \right\} + \text{const} \\ &= -\frac{\lambda}{4\pi\epsilon_0} \left\{ \frac{2x^2}{r_x(r_x + r_y)} + \frac{2y^2}{r_y(r_x + r_y)} \right\} + \text{const} \end{aligned}$$

$$\boxed{\phi = -\frac{\lambda}{2\pi\epsilon_0} \left\{ \frac{x^2}{r_x(r_x + r_y)} + \frac{y^2}{r_y(r_x + r_y)} \right\} + \text{const}}$$

- This formula agrees with the simple case of an axisymmetric beam with $r_x = r_y = r_b$
  - Discussed further in a simple homework problem

A7

2) Indirect Proof:

- More efficient method
- Steps useful for other constructions including moment calculations
  - See: J.J. Barnard, Introductory Lectures

Density has elliptical symmetry:

$$n(x,y) = n\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right) \qquad \text{function } n(\text{argument}) \text{ arbitrary}$$

The solution to the 2D Poisson equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{qn}{\epsilon_0}$$

in free-space is then given by

$$\phi = -\frac{q r_x r_y}{4\epsilon_0}\int_0^\infty d\xi\, \frac{\eta(\chi)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}} \qquad \chi \equiv \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi}$$

where $\eta(\chi)$ is a function defined such that

$$n(x,y) = \left.\frac{d\eta(\chi)}{d\chi}\right|_{\xi=0}$$

- Can show that a choice of η realizable for any elliptical symmetry *n*

A8

Prove that the solution is valid by direct substitution

$$\chi = \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} \quad \Longrightarrow \quad \begin{array}{ll} \dfrac{\partial \chi}{\partial x} = \dfrac{2x}{r_x^2+\xi} & \dfrac{\partial^2 \chi}{\partial x^2} = \dfrac{2}{r_x^2+\xi} \\ \dfrac{\partial \chi}{\partial y} = \dfrac{2y}{r_y^2+\xi} & \dfrac{\partial^2 \chi}{\partial y^2} = \dfrac{2}{r_y^2+\xi} \end{array}$$

Substitute in Poisson's equation, use the chain rule, and apply results above:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi =$$

$$-\frac{q r_x r_y}{4\epsilon_0}\int_0^\infty d\xi\, \frac{\left(\frac{d^2\eta}{d\chi^2}\right)\left(\frac{4x^2}{(r_x^2+\xi)^2} + \frac{4y^2}{(r_y^2+\xi)^2}\right) + \left(\frac{d\eta}{d\chi}\right)\left(\frac{2}{r_x^2+\xi} + \frac{2}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

Note:

$$d\chi = -\left[\frac{x^2}{(r_x^2+\xi)^2} + \frac{y^2}{(r_y^2+\xi)^2}\right] d\xi$$

Using this result the first integral becomes:

$$\int_0^\infty d\xi\, \frac{\left(\frac{d^2\eta}{d\chi^2}\right)\left(\frac{4x^2}{(r_x^2+\xi)^2} + \frac{4y^2}{(r_y^2+\xi)^2}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}} = -4\int_0^\infty d\xi\, \frac{\frac{d\eta^2}{d\chi^2}\frac{d\chi}{d\xi}}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

A9

Apply partial integration:

$$-4\int_0^\infty d\xi\, \frac{\frac{d\eta^2}{d\chi^2}\frac{d\chi}{d\xi}}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}} = -4\int_0^\infty d\xi\, \frac{\frac{d}{d\xi}\left(\frac{d\eta}{d\chi}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

$$= -4\int_0^\infty d\xi\, \frac{d}{d\xi}\left[\frac{\frac{d\eta}{d\chi}}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}\right] + 4\int_0^\infty d\xi\, \frac{d\eta}{d\chi}\frac{d}{d\xi}\frac{1}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

$$= -4\left.\frac{\frac{d\eta}{d\chi}}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}\right|_{\xi=0}^{\xi\to\infty} - 2\int_0^\infty d\xi\, \frac{\frac{d\eta}{d\chi}\left(\frac{1}{r_x^2+\xi}+\frac{1}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

in first term, upper limit vanishes since denominator ~ $\xi \to \infty$

$$= \frac{4}{r_x r_y}\left.\frac{d\eta}{d\chi}\right|_{\xi=0} - \boxed{2\int_0^\infty d\xi\, \frac{\frac{d\eta}{d\chi}\left(\frac{1}{r_x^2+\xi}+\frac{1}{r_y^2+\xi}\right)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}}$$

Term cancels
2nd integral

Giving:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\phi = -q\frac{r_x r_y}{4\epsilon_0}\frac{4}{r_x r_y}\left.\frac{d\eta(\chi)}{d\chi}\right|_{\xi=0} = -\frac{q}{\epsilon_0}n(x,y)$$

$$d\eta(\chi)/d\chi|_{\xi=0} = n(x,y)$$

Which verifies the ansatz.

A10

For a uniform density ellipse, we take:

$$\boxed{\eta(\chi) = \frac{\lambda}{q\pi r_x r_y}\begin{cases}\chi, & \text{if } \chi < 1 \\ 1, & \text{if } \chi > 1\end{cases}} \quad \longrightarrow \quad \frac{d\eta(\chi)}{d\chi} = \begin{cases}\frac{\lambda}{q\pi r_x r_y}, & \text{if } \chi < 1 \\ 0, & \text{if } \chi > 1\end{cases}$$

Then

$$\left.\frac{d\eta(\chi)}{d\chi}\right|_{\xi=0} = \begin{cases}\frac{\lambda}{q\pi r_x r_y}, & \text{if } \chi|_{\xi=0} < 1 \\ 0, & \text{if } \chi|_{\xi=0} > 1\end{cases} = \begin{cases}\frac{\lambda}{q\pi r_x r_y}, & \text{if } x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & \text{if } x^2/r_x^2 + y^2/r_y^2 > 1\end{cases}$$

Therefore, for this choice of

$$\left.\frac{d\eta(\chi)}{d\chi}\right|_{\xi=0} = n(x,y) \quad \text{for a uniform density elliptical beam}$$
$$\text{with radii } r_x,\ r_y \text{ and density } \lambda/(q\pi r_x r_y)$$

Apply these results to calculate

$$\phi = -\frac{q r_x r_y}{4\epsilon_0}\int_0^\infty d\xi\, \frac{\eta(\chi)}{\sqrt{r_x^2+\xi}\sqrt{r_y^2+\xi}}$$

$$\chi = \frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} \quad \Longrightarrow \quad \text{if } \frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} < 1, \quad \text{then}$$

$$\chi < 1 \quad \text{for all} \quad 0 \le \xi < \infty$$

A11

Then:

$$\phi = -\frac{q r_x r_y}{4\epsilon_0} \int_0^\infty d\xi \; \frac{\lambda}{q\pi r_x r_y} \left[\frac{x^2}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}} + \frac{y^2}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}} \right]$$

Using Mathematica or integral tables

$$\int_0^\infty d\xi \; \frac{1}{(r_x^2+\xi)^{3/2}(r_y^2+\xi)^{1/2}} = \frac{2}{r_x(r_x+r_y)}$$

$$\int_0^\infty d\xi \; \frac{1}{(r_x^2+\xi)^{1/2}(r_y^2+\xi)^{3/2}} = \frac{2}{r_y(r_x+r_y)}$$

Showing that:

$$\boxed{\phi = -\frac{\lambda}{2\pi\epsilon_0} \left[\frac{x^2}{r_x(r_x+r_y)} + \frac{y^2}{r_y(r_x+r_y)} \right] \; + \; \text{const}}$$

since an overall constant can always be added to the potential (the integral had a reference choice $\phi(x=y=0)=0$ built in.

A12

The steps introduced in this proof can also be simply extended to show that

- For steps, see J.J. Barnard, Introductory Lectures

$$\langle x \frac{\partial \phi}{\partial x} \rangle_\perp = -\frac{\lambda}{4\pi\epsilon_0} \frac{r_x}{r_x + r_y}$$

$$\langle y \frac{\partial \phi}{\partial y} \rangle_\perp = -\frac{\lambda}{4\pi\epsilon_0} \frac{r_y}{r_x + r_y}$$

$$\lambda \equiv q \int d^2x_\perp \, n$$

$$r_x \equiv \langle x^2 \rangle_\perp^{1/2}$$

$$r_y \equiv \langle y^2 \rangle_\perp^{1/2}$$

for *any* elliptic symmetry density profile

$$n(x, y) = \text{func}\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} \right)$$

In the introductory lectures, these results were applied to show that the KV envelope equations with evolving emittances can be applied to elliptic symmetry beams.

- Result first shown by Sacherer, IEEE Trans. Nuc. Sci. 18, 1105 (1971)

A13

Appendix B: Canonical Transformation of the KV Distribution

The single-particle equations of motion:

$$x''(s) + \left\{ \kappa_x(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_x(s)} \right\} x(s) = 0$$

$$y''(s) + \left\{ \kappa_y(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_y(s)} \right\} y(s) = 0$$

can be derived from the Hamiltonian:

$$\boxed{\begin{aligned} H_\perp(x, y, x', y'; s) = &\frac{1}{2}x'^2 + \left[\kappa_x(s) + \frac{2Q}{r_x(s)[r_x(s) + r_y(s)]} \right] \frac{x^2}{2} \\ &+ \frac{1}{2}y'^2 + \left[\kappa_y(s) + \frac{2Q}{r_y(s)[r_x(s) + r_y(s)]} \right] \frac{y^2}{2} \end{aligned}}$$

using:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

B1

Perform a canonical transform to new variables X,Y, X',Y' using the generating function

$$\boxed{F_2(x,y,X',Y') = \frac{x}{w_x}\left[X' + \frac{1}{2}xw'_x\right] + \frac{y}{w_y}\left[y' + \frac{1}{2}yw'_y\right]}$$

Then we have from Canonical Transform theory (see: Goldstein, Classical Mechanics, 2nd Edition, 1980)

$$X = \frac{\partial F_2}{\partial X'} = \frac{x}{w_x} \qquad x' = \frac{\partial F_2}{\partial x} = \frac{1}{w_x}(X' + xw'_x)$$

$$Y = \frac{\partial F_2}{\partial Y'} = \frac{y}{w_y} \qquad y' = \frac{\partial F_2}{\partial y} = \frac{1}{w_y}(Y' + yw'_y)$$

which give

<u>Transform</u>

$$X = x/w_x \qquad X' = w_x x' - xw'_x$$
$$Y = y/w_y \qquad Y' = w_y y' - yw'_y$$

<u>Inverse Transform</u>

$$x = w_x X \qquad x' = X'/w_x + w'_x X$$
$$y = w_y Y \qquad y' = Y'/w_y + w'_y Y$$

B2

The structure of the canonical transform results in transformed equations of motion in proper canonical form:

$$\tilde{H}_\perp = H_\perp + \frac{\partial F_2}{\partial s} \qquad \tilde{H}_\perp = \tilde{H}_\perp(X, Y, X', Y'; s)$$

$$\boxed{\begin{aligned} &\tilde{H} = \frac{1}{2w_x^2}X'^2 + \frac{1}{2w_y^2}Y'^2 + \frac{1}{2w_x^2}X^2 + \frac{1}{2w_y^2}Y^2 \\ &\frac{d}{ds}X = \frac{\partial \tilde{H}_\perp}{\partial X'} = \frac{X'}{w_x^2} \qquad \frac{d}{ds}X' = -\frac{\partial \tilde{H}_\perp}{\partial X} = -\frac{X}{w_x^2} \\ &\frac{d}{ds}Y = \frac{\partial \tilde{H}_\perp}{\partial Y'} = \frac{Y'}{w_y^2} \qquad \frac{d}{ds}Y' = -\frac{\partial \tilde{H}_\perp}{\partial Y} = -\frac{Y}{w_y^2} \end{aligned}}$$

- Caution: X' merely denotes the conjugate variable to X : $\frac{d}{ds}X \neq X'$
- X and X' both have dimensions (meters)^(1/2)
- Equations of motion can be verified directly from transform equations (see problem sets)
- Transformed Hamiltonian $\tilde{H}_\perp$ is explicitly s dependent due to *w_x* and *w_y* lattice functions

B3

Following Davidson (Physics of Nonneutral Plasmas), the equations of motion

$$\frac{d}{ds}X' + \frac{1}{w_x^2}X = 0 \qquad \frac{d}{ds}X' = -\frac{X}{w_x^2}$$

$$\frac{d}{ds}Y' + \frac{1}{w_y^2}Y = 0 \qquad \frac{d}{ds}Y' = -\frac{Y}{w_y^2}$$

have a psudo-harmonic oscillator solution

$$X(s) = X_i \cos\psi_x(s) + X_i' \sin\psi_x(s)$$

$$\psi_x(s) = \int_{s_i}^{s} \frac{d\tilde{s}}{w_x^2(\tilde{s})} \qquad \begin{aligned} X_i &= \text{const} \\ X_i' &= \text{const} \end{aligned}$$ set by initial conditions

This explicitly verifies the simple, symmetrical form of the Courant-Snyder invariants in the transformed variables:

$$\boxed{\begin{aligned} X^2 + X'^2 &= \left(\frac{x}{w_x}\right)^2 + (w_x x' - x w_x')^2 = \text{const} \\ Y^2 + Y'^2 &= \left(\frac{y}{w_y}\right)^2 + (w_y y' - y w_y')^2 = \text{const} \end{aligned}}$$

B4

The canonical transforms render the KV distribution much simpler to express. First examine how phase-space areas transform:

$$\begin{aligned} dxdy &= w_x w_y dX dY \\ dx'dy' &= \frac{dX'dY'}{w_x w_y} \end{aligned} \qquad \Longrightarrow \qquad dxdydx'dy' = dXdYdX'dY'$$

- The property *dx dy dx' dy'* = *dX dY dX' dY'* is a consequence of canonical transforms preserving phase-space area

Because phase space area is conserved, the distribution in transformed phase-space variables is identical to the original distribution. Therefore, for the KV distribution

$$\begin{aligned} f_\perp &= \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y}\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right] \\ &= \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y}\delta\left[\frac{X^2 + X'^2}{\varepsilon_x} + \frac{Y^2 + Y'^2}{\varepsilon_y} - 1\right] \qquad r_x = \sqrt{\varepsilon_x} w_x \end{aligned}$$

- Transformed form simpler and more symmetrical
- Exploited to simplify calculation of distribution moments and projections

B5

Density Calculation:

As a first example application of the canonical transform, prove that the density projection of the KV distribution is a uniform density ellipse. Doing so will prove the consistency of the KV equilibrium:

- If density projection is as assumed then the Courant-Snyder invariants are valid
- Steps used can be applied to calculate other moments/projections
- Steps can be applied to continuous focusing without using the transformations

$$n(x,y) = \int dx'dy' \; f_\perp = \int \frac{dX'dY'}{w_x w_y} f_\perp$$

$$r_x = \sqrt{\varepsilon_x} w_x \qquad U_x = X'/\sqrt{\varepsilon_x}$$

$$r_y = \sqrt{\varepsilon_y} w_y \qquad U_y = Y'/\sqrt{\varepsilon_y}$$

$$dU_x dU_y = \frac{dX'dY'}{\sqrt{\varepsilon_x \varepsilon_y}}$$

$$\boxed{n = \frac{\lambda}{q\pi^2 r_x r_y} \int dU_x dU_y \; \delta\left[U_x^2 + U_y^2 - \left(1 - \frac{X^2}{\varepsilon_x} - \frac{Y^2}{\varepsilon_y}\right)\right]}$$

B6

Exploit the cylindrical symmetry

$$U_\perp^2 = U_x^2 + U_y^2 \qquad dU_x dU_y = d\psi U_\perp dU_\perp = d\psi \frac{dU_\perp^2}{2}$$

$$n(x,y) = \frac{\lambda}{q\pi^2 r_x r_y} \int_{-\pi}^{\pi} d\psi \int_0^\infty \frac{dU_\perp^2}{2}\, \delta\left[U_\perp^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right]$$

giving

$$\boxed{\begin{aligned} n(x,y) &= \frac{\lambda}{q\pi r_x r_y} \int_0^\infty dU_\perp^2\, \delta\left[U_\perp^2 - \left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] \\ &= \begin{cases} \frac{\lambda}{q\pi r_x r_y} = \hat{n}, & \text{if } x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & \text{if } x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases} \end{aligned}}$$

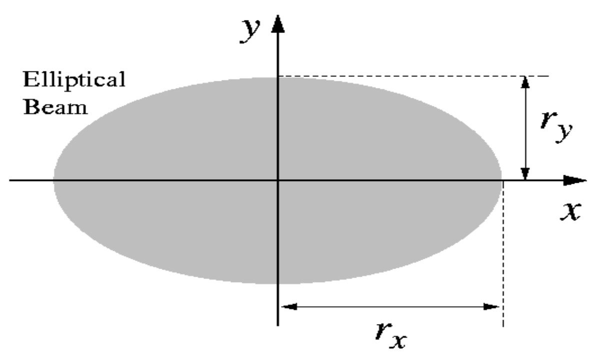
Shows that the singular KV distribution yields the required uniform density elliptical projection required for self-consistency!

Note:

$$\hat{n} = \frac{\lambda}{q\pi r_x r_y}$$

Line Charge: $\lambda = \text{const}$

Area Ellipse = $\pi r_x r_y$



B7

// Aside

An interesting footnote to this Appendix is that an infinity of canonical generating functions can be applied to transform the KV distribution in standard quadratic form

$$f_\perp \sim \delta[X^2 + X'^2 + Y^2 + Y'^2 - \text{const}]$$

to other sets of variables. These distributions have underlying KV form.

- Not logical to label transformed KV distributions as "new" but this has been done in the literature
  - Could generate an infinity of KV like equilibria in this manner
- Identifying specific transforms with physical relevance can be useful even if the canonical structure of the distribution is still KV
  - Helps identify basic design criteria with envelope consistency equations etc.
  - Example of this is a self-consistent KV distribution formulated for quadrupole skew coupling

//

B8

S4: Continuous Focusing limit of the KV Equilibrium Distribution

Continuous focusing, axisymmetric beam

$$\boxed{\begin{aligned} &\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const} \\ &\varepsilon_x = \varepsilon_y \equiv \varepsilon \\ &r_x = r_y \equiv r_b \end{aligned}}$$

Undepressed betatron wavenumber

KV envelope equation

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

immediately reduces to:

$$r_b'' + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

with solution

$$\boxed{r_b = \left(\frac{Q + \sqrt{4k_{\beta 0}^2 \varepsilon^2 + Q^2}}{2k_{\beta 0}^2} \right)^{1/2} = \text{const}}$$

Similarly, the particle equations of motion within the beam are:

$$x'' + \left\{ \kappa_x - \frac{2Q}{[r_x + r_y] r_x} \right\} x = 0$$

$$y'' + \left\{ \kappa_y - \frac{2Q}{[r_x + r_y] r_y} \right\} y = 0$$

reduce to

$$\mathbf{x}''_\perp + k_\beta^2 \mathbf{x}_\perp = 0 \qquad \boxed{k_\beta \equiv \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} = \text{const}}$$

Depressed betatron wavenumber

with solution

$$\boxed{\mathbf{x}_\perp(s) = \mathbf{x}_{\perp i} \cos[k_\beta(s - s_i)] + \frac{\mathbf{x}'_{\perp i}}{k_\beta} \sin[k_\beta(s - s_i)]}$$

Space-charge tune depression (rate of phase advance same everywhere, L_p arb.)

$$\frac{k_\beta}{k_{\beta 0}} = \frac{\sigma}{\sigma_0} = \left(1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right)^{1/2}$$

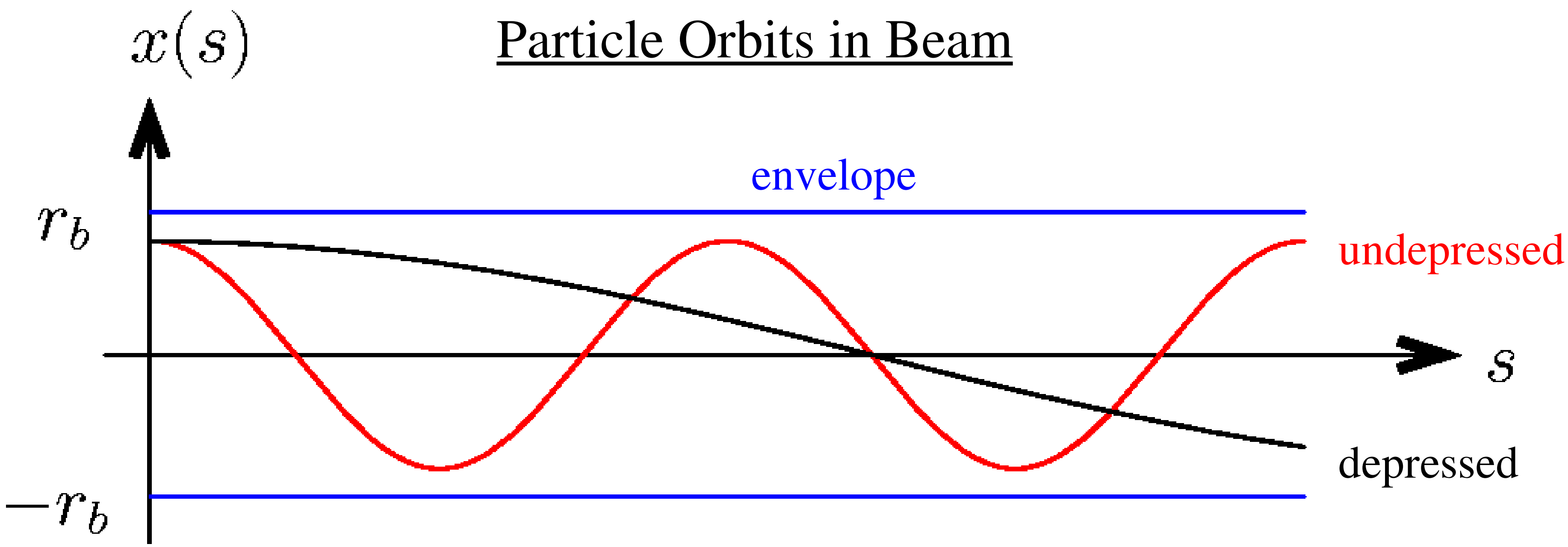
$$\begin{array}{ccccc} 0 & \leq & \dfrac{\sigma}{\sigma_0} & \leq & 1 \\ \varepsilon \rightarrow 0 & & & & Q \rightarrow 0 \end{array}$$

$$\Rightarrow \quad k_{\beta 0}^2 r_b^2 = Q$$

envelope equation

Continuous Focusing KV Equilibrium –
Undepressed and depressed particle orbits in the *x*-plane

$$k_\beta = \frac{\sigma}{\sigma_0} k_{\beta 0} \qquad \frac{\sigma}{\sigma_0} = 0.2 \qquad y = 0 = y'$$



Much simpler in details than the periodic focusing case,
but qualitatively similar in that space-charge "depresses" the
rate of particle phase advance

Continuous Focusing KV Beam – Equilibrium Distribution Form

Using

$$\lambda = q\pi\hat{n}r_b^2 \qquad \hat{n} = \text{const} \qquad \text{density within the beam}$$

for the beam line charge and

$$\delta(\text{const}\cdot x) = \frac{\delta(x)}{\text{const}}$$

the full elliptic beam KV distribution can be expressed as :

- See next slide for steps involved in the form reduction

$$\boxed{\begin{aligned} f_\perp &= \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r'_x x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r'_y y}{\varepsilon_y}\right)^2 - 1\right] \\ &= \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b}) \end{aligned}}$$

where

$$\begin{aligned} H_\perp &= \frac{1}{2}\mathbf{x}'^2_\perp + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}^2_\perp \\ &= \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k_{\beta 0}^2\mathbf{x}^2_\perp + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \end{aligned}$$

-- Hamiltonian (on-axis value 0 ref)

$$H_{\perp b} \equiv \frac{\varepsilon^2}{2r_b^2} = \text{const}$$

-- Hamiltonian at beam edge

/// Aside: Steps of derivation

Using:

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon \qquad\qquad \lambda = q\pi\hat{n}r_b^2 = \text{const}$$

$$r_x = r_y \equiv r_b = \text{const}$$

$$f_\perp = \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 - 1\right]$$

$$= \frac{\hat{n}r_b^2}{\pi\varepsilon^2}\delta\left(\frac{x^2}{r_b^2} + \frac{y^2}{r_b^2} + \frac{r_b^2 x'^2}{\varepsilon^2} + \frac{r_b^2 y'^2}{\varepsilon^2} - 1\right)$$

Using:

$$\delta(\text{const}\cdot x) = \frac{\delta(x)}{\text{const}}$$

$$f_\perp = \frac{\hat{n}}{2\pi}\delta\left(\frac{1}{2}\mathbf{x}_\perp'^2 + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}_\perp^2 - \frac{\varepsilon^2}{2r_b^2}\right)$$

The solution for the potential for the uniform density beam *inside* the beam is:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} = -\frac{\lambda}{\pi\epsilon_0 r_b^2} \quad\longrightarrow\quad \phi = -\frac{\lambda}{4\pi\epsilon_0 r_b^2}\mathbf{x}_\perp^2 + \text{const}$$

The Hamiltonian becomes:

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$=\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp - \frac{q\lambda}{4\pi m\gamma_b^3\beta_b^2c^2}\mathbf{x}^2_\perp + \text{const}$$

$$=\frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp - \frac{Q}{2r_b^2}\mathbf{x}^2_\perp + \text{const}$$

$$Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} = \text{const}$$

From the equilibrium envelope equation:

$$k^2_{\beta 0} = \frac{Q}{r_b^2} + \frac{\varepsilon^2}{r_b^4}$$

The Hamiltonian reduces to:

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}^2_\perp + \text{const}$$

with edge value (turning point with zero angle):

$$H_{\perp b} \equiv \frac{\varepsilon^2}{2r_b^2} + \text{const}$$

Giving (constants are same in Hamiltonian and edge value and subtract out):

$$f_\perp = \frac{\hat{n}}{2\pi}\delta\left(\frac{1}{2}\mathbf{x}'^2_\perp + \frac{\varepsilon^2}{2r_b^4}\mathbf{x}^2_\perp - \frac{\varepsilon^2}{2r_b^2}\right) = \frac{\hat{n}}{2\pi}\delta\left(H_\perp - H_{\perp b}\right)$$

///

Equilibrium distribution

$$\boxed{f_\perp(H_\perp) = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})}$$

$$H_{\perp b} = \frac{\varepsilon^2}{2r_b^2} = \text{const}$$

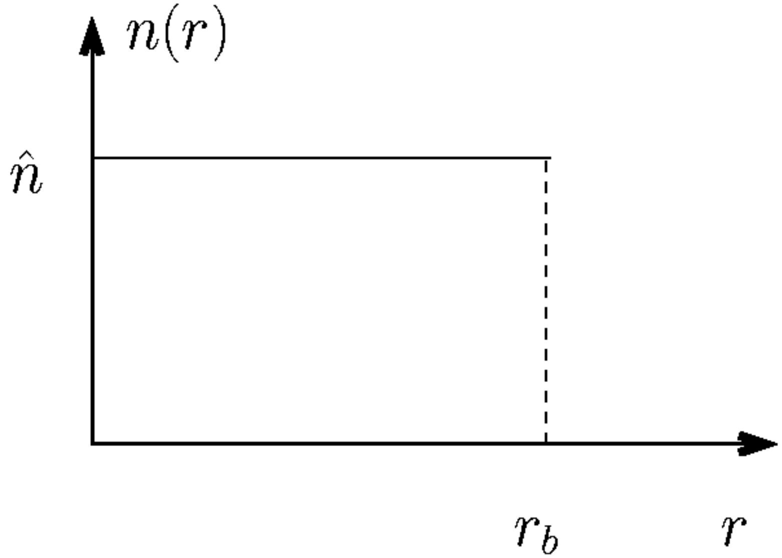
$$\hat{n} = \text{const}$$

then it is straightforward to explicitly calculate (see homework problems)

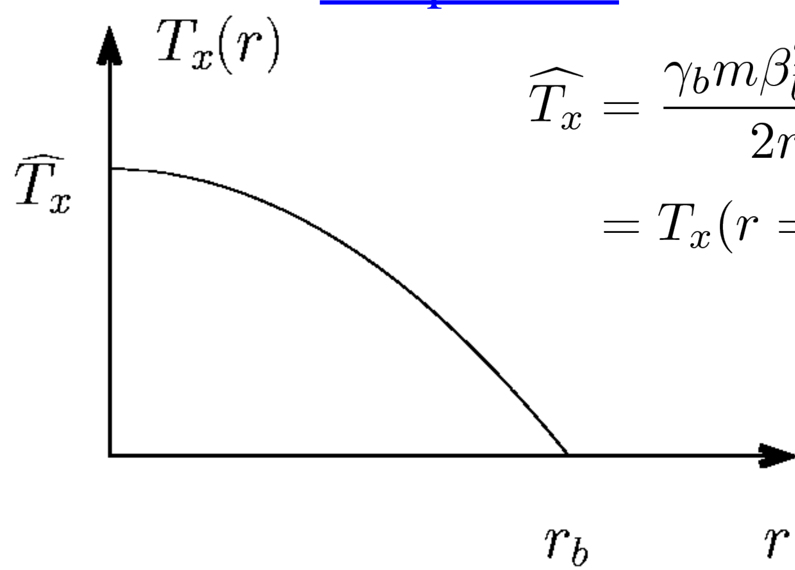
Density:
$$n = \int d^2x'_\perp \; f_\perp = \begin{cases} \hat{n}, & 0 \leq r < r_b \\ 0, & r_b < r \end{cases}$$

Temperature:
$$T_x = \gamma_b m \beta_b^2 c^2 \frac{\int d^2x'_\perp \; x'^2 f_\perp}{\int d^2x'_\perp \; f_\perp} = \begin{cases} \widehat{T_x}(1 - r^2/r_b^2), & 0 \leq r < r_b \\ 0, & r_b < r \end{cases}$$

Density



Temperature



$$\widehat{T_x} = \frac{\gamma_b m \beta_b^2 c^2 \varepsilon^2}{2r_b^2}$$
$$= T_x(r=0)$$

Continuous Focusing KV Beam – Comments

For continuous focusing, $H_\perp$ is a single particle constant of the motion (see problem sets), so it is not surprising that the KV equilibrium form reduces to a delta function form of $f_\perp(H_\perp)$

- Because of the delta-function distribution form, all particles in the continuous focusing KV beam have the same transverse energy with $H_\perp = H_{\perp b} = \text{const}$

Several textbook treatments of the KV distribution derive continuous focusing versions and then just write down (if at all) the periodic focusing version based on Courant-Snyder invariants. This can create a false impression that the KV distribution is a Hamiltonian-type invariant in the general form.

- For non-continuous focusing channels there is no simple relation between Courant-Snyder type invariants and $H_\perp$

S5: Stationary Equilibrium Distributions in Continuous Focusing Channels

Take

$$\boxed{\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}}$$

- Real transport channels have s-varying focusing functions
- For a rough correspondence to physical lattices take: $k_{\beta 0} = \sigma_0 / L_p$

A valid family of equilibria can be constructed for any choice of function:

$$\boxed{f_\perp = f_\perp(H_\perp) \geq 0} \qquad H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3 \beta_b^2 c^2}$$

ϕ must be calculated consistently from the (generally nonlinear) Poisson equation:

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\epsilon_0}\int d^2x_\perp' \; f_\perp(H_\perp)}$$

- Solutions generated will be steady-state $(\partial/\partial s = 0)$
- When $f_\perp = f_\perp(H_\perp)$, the Poisson equation *only* has axisymmetric solutions with $\partial/\partial\theta = 0$ [see: Lund, PRSTAB **10**, 064203 (2007)]

The Hamiltonian is only equivalent to the Courant-Snyder invariant in continuous focusing (see: Transverse Particle Equations). In periodic focusing channels $\kappa_x(s)$ and $\kappa_y(s)$ vary in s and the Hamiltonian is *not* a constant of the motion.

The axisymmetric Poisson equation simplifies to:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = -\frac{qn}{\epsilon_0} = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; f_\perp(H_\perp)$$

For notational convenience, introduce an effective (add applied component and rescale) potential defined by

$$\boxed{\psi(r) \equiv \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^2}} \qquad r = \sqrt{x^2+y^2}$$

then

$$\boxed{H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \psi}$$

and system axisymmetry can be exploited to calculate the beam density (see earlier aside slides on integral symmetries for steps) as:

$$\boxed{n(r) = \int d^2x'_\perp \; f_\perp(H_\perp) \; = 2\pi\int_\psi^\infty dH_\perp \; f_\perp(H_\perp)}$$

The Poisson equation can then be expressed in terms of the stream function as:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = 2k_{\beta 0}^2 - \frac{2\pi q^2}{m\epsilon_0\gamma_b^3\beta_b^2 c^2}\int_{\psi(r)}^\infty dH_\perp \; f_\perp(H_\perp)}$$

To characterize a choice of equilibrium function $f_\perp(H_\perp)$, the (transformed) Poisson equation must be solved

- Equation is, in general, *highly* nonlinear rendering the procedure difficult

Some general features of equilibria can still be understood:

- Apply rms equivalent beam picture and interpret in terms of moments
- Calculate equilibria for a few types of very different functions to understand the likely range of characteristics

Moment properties of continuous focusing equilibrium distributions

Equilibria with *any* valid equilibrium $f_\perp(H_\perp)$ satisfy the
rms equivalent beam matched beam envelope equation:

$$\boxed{k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0}$$

- Describes average radial force balance of particles
- Uses the result (see J.J. Barnard, Intro. Lectures): $\langle x \partial\phi/\partial x\rangle_\perp = -\lambda/(8\pi\epsilon_0)$

where

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const} \qquad \lambda = q\int d^2x_\perp \int d^2x'_\perp \; f_\perp(H_\perp)$$

$$r_b^2 = 2\langle r^2\rangle_\perp = \frac{\int_0^\infty dr\; r^3 \int_\psi^\infty dH_\perp \; f_\perp(H_\perp)}{\int_0^\infty dr\; r \int_\psi^\infty dH_\perp \; f_\perp(H_\perp)}$$

$$\varepsilon^2 = 2r_b^2 \langle \mathbf{x}'^2_\perp\rangle_\perp = 2r_b^2 \frac{\int_0^\infty dr\; r \int_\psi^\infty dH_\perp \; (H_\perp - \psi) f_\perp(H_\perp)}{\int_0^\infty dr\; r \int_\psi^\infty dH_\perp \; f_\perp(H_\perp)}$$

$$\langle \cdots \rangle_\perp = \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp(H_\perp)}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp(H_\perp)}$$

Parameters used to define the equilibrium function

$$f_\perp(H_\perp)$$

should be cast in terms of

$$Q, \;\; \varepsilon, \;\; r_b$$

for use in accelerator applications. The rms equivalent beam equations can be used to carry out needed parameter eliminations. Such eliminations can be highly nontrivial due to the nonlinear form of the equations.

A kinetic temperature can also be calculated

$$T_x = \langle x'^2 \rangle_{\mathbf{x}'_\perp} \qquad\qquad \langle \cdots \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \; \cdots f_\perp}{\int d^2x'_\perp \; f_\perp}$$

$$\boxed{n(r)T_x(r) = \frac{1}{2}\int d^2x'_\perp \; \mathbf{x}'^2_\perp f_\perp(H_\perp) \; = 2\pi \int_\psi^\infty dH_\perp \; (H_\perp - \psi) f_\perp(H_\perp)}$$

which is also related to the emittance,

$$\langle x'^2 \rangle_\perp = \frac{\int d^2x_\perp \; nT_x}{\int d^2x_\perp n} \qquad\qquad \boxed{\varepsilon^2 = 16\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp = 4r_b^2 \frac{\int d^2x_\perp \; nT}{\int d^2x_\perp \; n}}$$

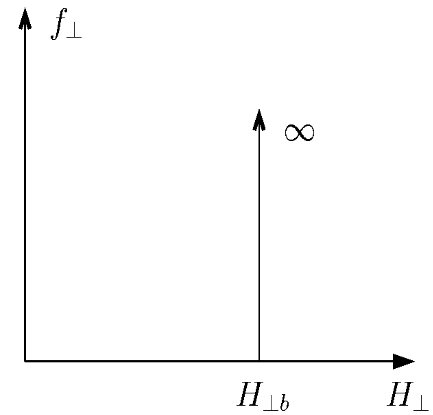
Choices of continuous focusing equilibrium distributions:

Common choices for $f_\perp(H_\perp)$ analyzed in the literature:

1) KV (already covered)

$$\boxed{f_\perp \propto \delta(H_\perp - H_{\perp b})}$$

$$H_{\perp b} = \text{const}$$

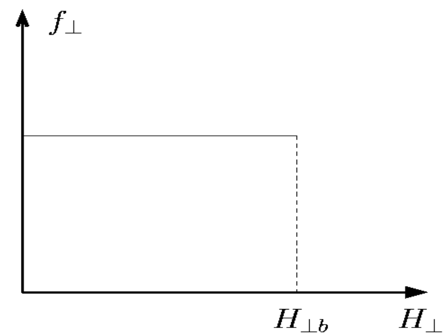


2) Waterbag (to be covered)

[see M. Reiser, *Charged Particle Beams*, (1994, 2008)]

$$\boxed{f_\perp \propto \Theta(H_{\perp b} - H_\perp)}$$

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x \end{cases}$$

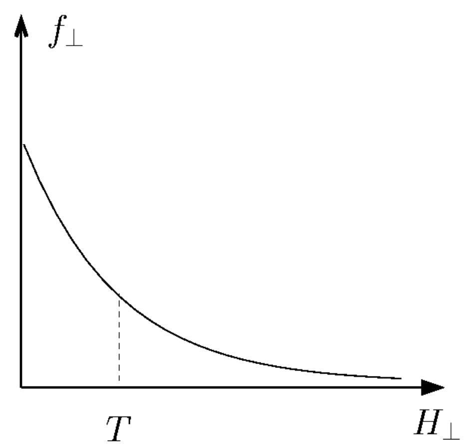


3) Thermal (to be covered)

[see M. Reiser; Davidson, *Noneutral Plasmas*, 1990]

$$\boxed{f_\perp \propto \exp(-H_\perp/T)}$$

$$T = \text{const} > 0$$



Infinity of choices can be made for an infinity of papers!

- Fortunately, range of behavior can be understood with a few reasonable choices

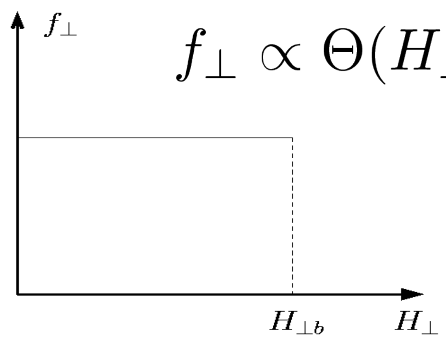
Preview of what we will find: When relative space-charge is strong, all smooth equilibrium distributions expected to look similar

Constant charge and focusing: $Q = 10^{-4} \qquad k_{\beta 0}^2 = \text{const}$

Vary relative space-charge strength: $\sigma/\sigma_0 = 0.1,\ 0.2,\ \cdots,\ 0.9$

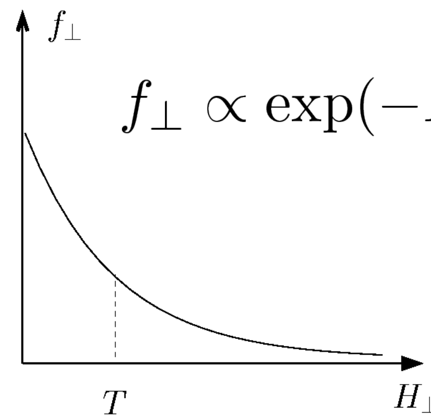
Waterbag Distribution

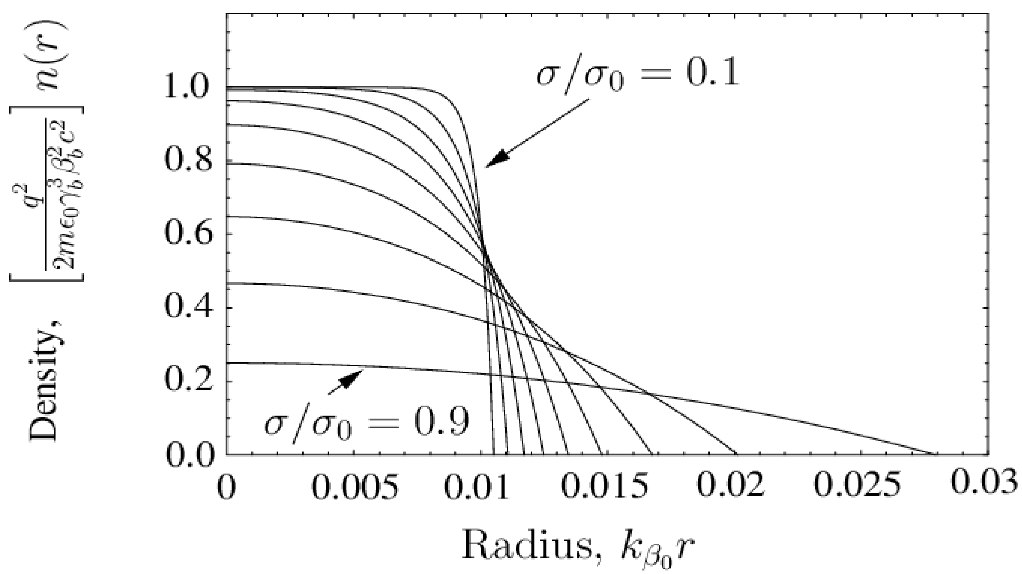
$$f_\perp \propto \Theta(H_{\perp b} - H_\perp)$$

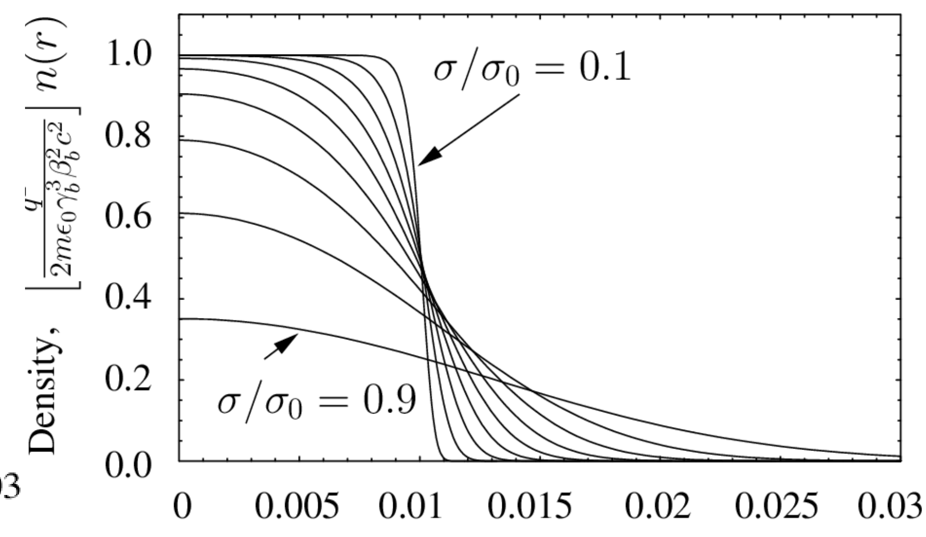


Thermal Distribution

$$f_\perp \propto \exp(-H_\perp/T)$$







Edge shape varies with distribution choice, but cores similar when σ/σ_0 small

S6: Continuous Focusing: The Waterbag Equilibrium Distribution:

[Reiser, *Theory and Design of Charged Particle Beams*, Wiley (1994, 2008); and Review: Lund, Kikuchi, and Davidson, PRSTAB, to be published (2008)]

Waterbag distribution:

$$\boxed{f_\perp(H_\perp) = f_0 \Theta(H_b - H_\perp) \qquad \Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}}$$

$f_0 = \text{const}$

$H_b = \text{const}$ Edge Hamiltonian

The physical edge radius r_e of the beam will be related to the edge Hamiltonian:

$$\boxed{H_\perp|_{r=r_e} = H_b}$$

Note (generally): $r_e \neq r_b \equiv 2\langle x^2\rangle_\perp^{1/2}$

$r_e > r_b$

Using previous formulas the equilibrium density can then be calculated as:

$$H_\perp = \mathbf{x}'^2_\perp/2 + \psi \qquad \psi = k_{\beta 0}^2 r^2/2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^2}$$

$$\boxed{n(r) = \int d^2x'_\perp \; f_\perp = 2\pi f_0 \begin{cases} H_b - \psi(r), & \psi < H_b, \\ 0, & \psi > H_b. \end{cases}}$$

The Poisson equation of the equilibrium can be expressed within the beam $(r < r_e)$ as:

$$\boxed{\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - k_0^2\psi = 2k_{\beta 0}^2 - k_0^2 H_b \\ &k_0^2 \equiv \frac{2\pi q^2 f_0}{\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const} \end{aligned}}$$

This is a modified Bessel function equation and the solution within the beam regular at the origin r = 0 and satisfying $\psi(r = r_e) = H_b$ is given by

$$\boxed{\psi(r) = H_b - 2\frac{k_{\beta 0}^2}{k_0^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right]}$$

where $I_\ell(x)$ is a modified Bessel function of order ℓ

The density is then expressible within the beam $(r < r_e)$ as:

$$\boxed{\begin{aligned} n(r) &= 4\pi f_0 \frac{k_{\beta 0}^2}{k_0^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right] \\ &= \frac{2\epsilon_0 m \gamma_b^2 \beta_b^2 c^2 k_{\beta 0}^2}{q^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right] \end{aligned}}$$

Similarly, the local beam temperature within the beam can be calculated as:

$$\boxed{\begin{aligned} T_x(r) = \langle x'^2 \rangle_{\mathbf{x}'_\perp} &= \frac{k_{\beta 0}^2}{k_0^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right] \\ &\propto n(r) \end{aligned}}$$

The proportionality between the temperature *T_x*(*r*) and the density *n*(*r*) is a consequence of the waterbag equilibrium distribution choice and is *not* a general feature of continuous focusing.

The waterbag distribution expression can now be expressed as:

$$\boxed{f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp) = f_0 \Theta\left(2\frac{k_{\beta 0}^2}{k_0^2}\left[1 - \frac{I_0(k_0 r)}{I_0(k_0 r_e)}\right] - \frac{1}{2}\mathbf{x}'^2_\perp\right)}$$

- The edge Hamiltonian value H_b has been eliminated
- Parameters are:

f_0 .... distribution normalization

$k_0 r_e$ .... scaled edge radius

$k_{\beta 0}/k_0$ .... scaled focusing strength

Parameters preferred for accelerator applications:

$$k_{\beta 0}, \quad Q, \quad \varepsilon_x = \varepsilon_y = \varepsilon_b$$

Needed constraints to eliminate parameters in terms of our preferred set will now be derived.

Parameters constraints for the waterbag equilibrium beam

First calculate the beam line-charge:

$$\lambda = 2\pi q \int_0^{r_e} dr\ rn(r) = 4\pi^2 q f_0 \frac{k_{\beta 0}^2}{k_0^2} r_e^2 \left[1 - \frac{2}{k_0 r_e} \frac{I_1(k_0 r_e)}{I_0(k_0 r_e)}\right]$$

$$\boxed{\lambda = 2\pi q \int_0^{r_e} dr\ rn(r) = 4\pi^2 q f_0 \frac{k_{\beta 0}^2}{k_0^2} r_e^2 \frac{I_2(k_0 r_e)}{I_0(k_0 r_e)}}$$

here we have employed the modified Bessel function identities (ℓ integer):

$$\frac{d}{dx}[x^\ell I_\ell(x)] = x^\ell I_{\ell-1}(x),$$

$$-\frac{2\ell}{x} I_\ell(x) = I_{\ell+1}(x) - I_{\ell-1}(x),$$

Similarly, the beam rms edge radius can be explicitly calculated as:

$$r_b^2 = 2\langle r^2 \rangle_\perp = 2\frac{\int_0^{r_e} dr\ r^3 n(r)}{\int_0^{r_e} dr\ rn(r)}$$

$$\boxed{\left(\frac{r_b}{r_e}\right)^2 = \frac{I_0(k_0 r_e)}{I_2(k_0 r_e)} - \frac{4}{(k_0 r_e)^2}\left[2 + (k_0 r_e)\frac{I_3(k_0 r_e)}{I_2(k_0 r_e)}\right]}$$

The perveance is then calculated as:

$$\boxed{Q \equiv \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2} = (k_{\beta 0} r_e)^2 \frac{I_2(k_0 r_e)}{I_0(k_0 r_e)}}$$

The edge and perveance equations can then be combined to obtain a parameter constraint relating $k_0 r_e$ to desired system parameters:

$$\boxed{\frac{k_{\beta 0}^2 r_b^2}{Q} = \frac{I_0^2(k_0 r_e)}{I_2^2(k_0 r_e)} - \frac{4}{(k_0 r_e)^2}\left[2\frac{I_0(k_0 r_e)}{I_2(k_0 r_e)} + (k_0 r_e)\frac{I_0(k_0 r_e) I_3(k_0 r_e)}{I_2^2(k_0 r_e)}\right]}$$

Here, any of the 3 system parameters on the LHS may be eliminated using the matched beam envelope equation to effect alternative parameterizations:

$$k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_b^2}{r_b^3} = 0 \longrightarrow \text{eliminate any of:} \quad k_{\beta 0}^2, \quad r_b \quad Q$$

The rms equivalent beam concept can also be applied to show that:

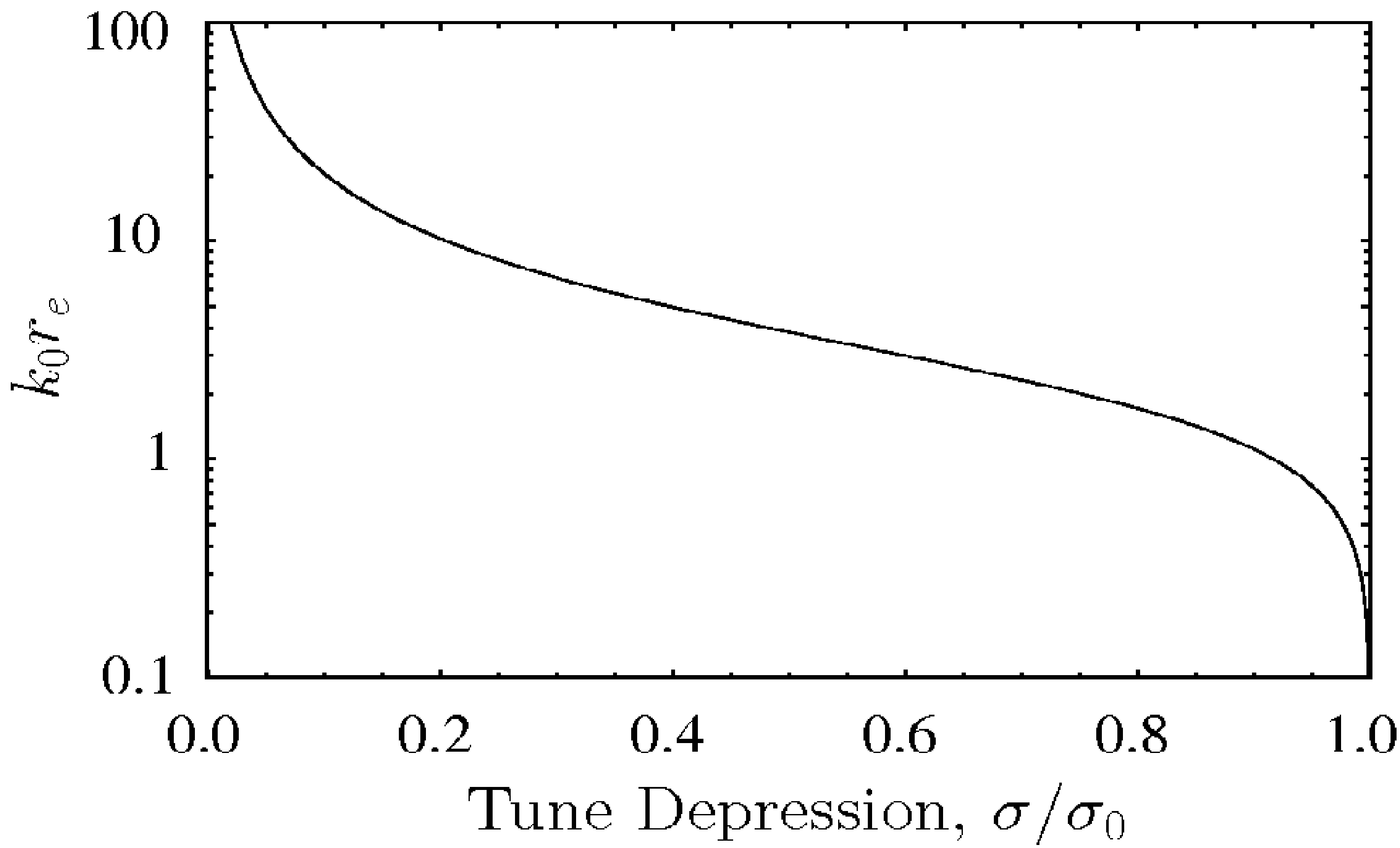
$$\boxed{\frac{k_{\beta 0}^2 r_b^2}{Q} = \frac{1}{1 - (\sigma/\sigma_0)^2}}$$

rms equivalent KV measure of σ/σ_0

- Space-charge really nonlinear and the Waterbag equilibrium has a spectrum of σ

The constraint is plotted over the full range of effective space-charge strength:

$$\boxed{\frac{1}{1-(\sigma/\sigma_0)^2} = \frac{I_0^2(k_0 r_e)}{I_2^2(k_0 r_e)} - \frac{4}{(k_0 r_e)^2}\left[2\frac{I_0(k_0 r_e)}{I_2(k_0 r_e)} + (k_0 r_e)\frac{I_0(k_0 r_e) I_3(k_0 r_e)}{I_2^2(k_0 r_e)}\right]}$$



- Equilibrium parameter $k_0 r_e$ uniquely fixes effective space-charge strength

///Aside: Parameter choices and limits of the constraint equation

Some prefer to use an alternative space-charge strength measure to σ/σ_0 and use a so-called self-field parameter defined in terms of the on-axis plasma frequency of the distribution:

Self-field parameter:

$$s_b \equiv \frac{\hat{\omega}_p^2}{2\gamma_b^3\beta_b^2 c^2 k_{\beta 0}^2} \qquad \hat{\omega}_p^2 \equiv \frac{q^2 \hat{n}}{m\epsilon_0} \qquad \begin{aligned} \hat{n} &= n(r=0) \\ &= \text{on-axis plasma density} \end{aligned}$$

For a KV equilibrium, s_b and σ/σ_0 are simply related:

$$s_b = 1 - \left(\frac{\sigma}{\sigma_0}\right)^2$$

For a waterbag equilibrium, s_b and $k_0 r_e$ (from which σ/σ_0 can be calculated) are related by:

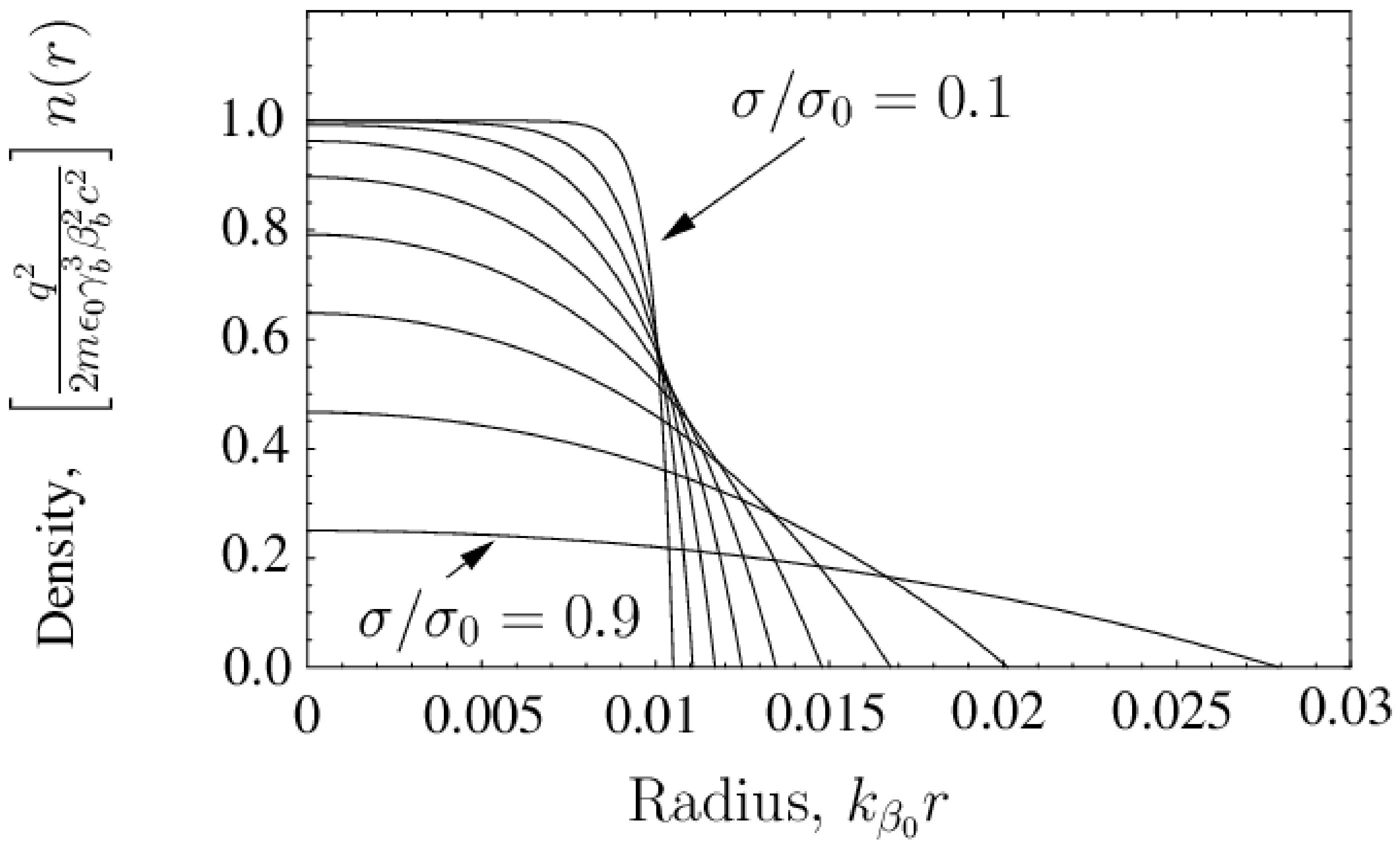
$$s_b = 1 - \frac{1}{I_0(k_0 r_e)}$$

Generally, for smooth (non-KV) equilibria, s_b turns out to be a logarithmically insensitive parameter for strong space-charge strength (see tables in S6 and S7) ///

Use parameter constraints to plot properties of waterbag equilibrium

1) Density and temperature profile at fixed line charge and focusing strength

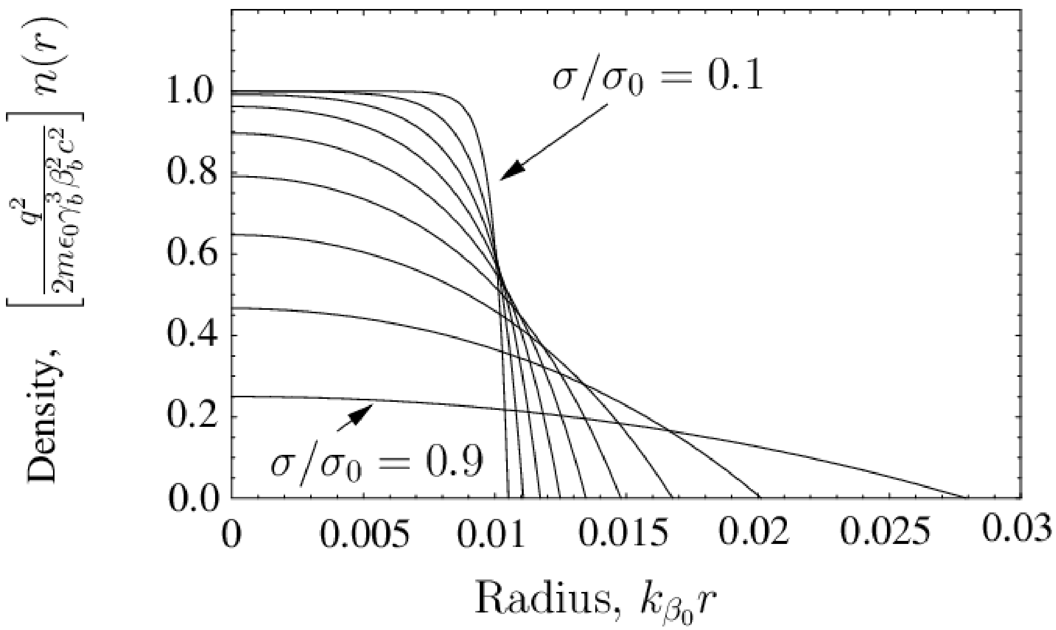
$$Q = 10^{-4} \qquad k_{\beta 0}^2 = \text{const}$$



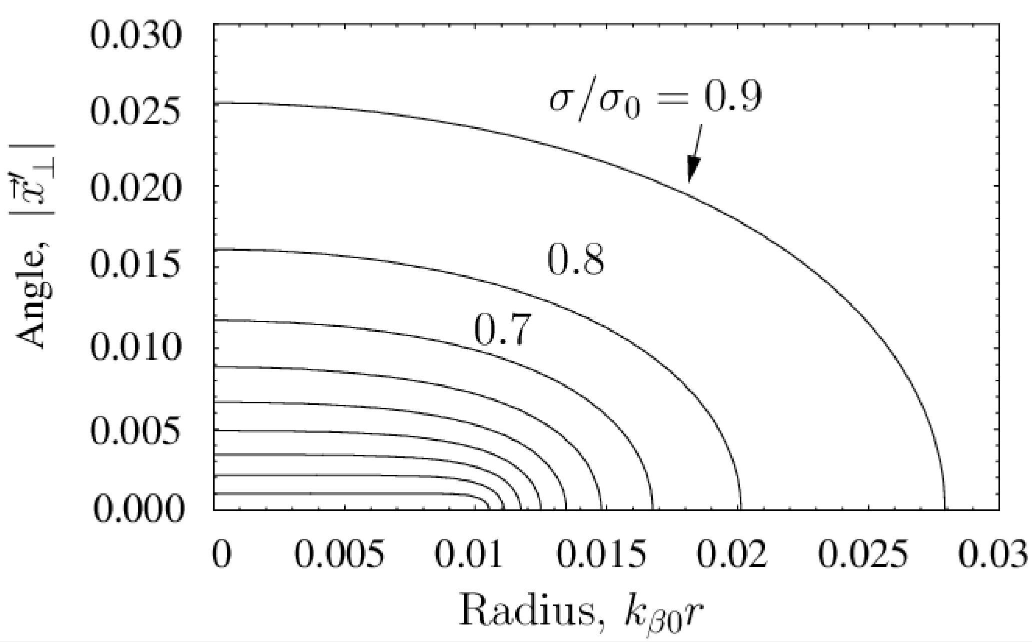
- Parabolic density for weak space-charge and flat in the core out to a sharp edge for strong space charge
- For the waterbag equilibrium, temperature $T(r)$ is proportional to density $n(r)$ so the same curves apply for $T(r)$

2) Phase-space boundary of distribution at fixed line charge and focusing strength

$Q = 10^{-4}$ $\qquad$ $k_{\beta 0}^2 = \text{const}$



Density Profile



Edge of distribution in phase-space

3) Summary of scaled parameters for example plots:

					$Q = 10^{-4}$	
σ/σ_0	s_b	$\frac{k_{\beta 0}^2 r_b^2}{Q}$	$k_0 r_e$	$\frac{r_e}{r_b}$	$\frac{k_0}{k_{\beta 0}}$	$10^3 \times k_{\beta 0}\varepsilon_b$
0.9	0.2502	5.263	1.112	1.217	39.81	0.4737
0.8	0.4666	2.778	1.709	1.208	84.87	0.2222
0.7	0.6477	1.961	2.304	1.197	137.5	0.1373
0.6	0.7916	1.563	2.979	1.183	201.5	0.09375
0.5	0.8968	1.333	3.821	1.166	283.8	0.06667
0.4	0.9626	1.190	4.978	1.144	398.7	0.04762
0.3	0.9928	1.099	6.789	1.118	579.3	0.03297
0.2	0.9997	1.042	10.25	1.085	925.6	0.02083
0.1	1.0000	1.010	20.38	1.046	1938.	0.01010

S7: Continuous Focusing: The Thermal Equilibrium Distribution:

[Davidson, Physics of Nonneutral Plasma, Addison Wesley (1990) and Reiser, Theory and Design of Charged Particle Beams, Wiley (1994, 2008)]

In an infinitely long continuous focusing channel, collisions will eventually relax the beam to thermal equilibrium. The Fokker-Planck equation predicts that the unique Maxwell-Boltzmann distribution describing this limit is:

$$\lim_{s\to\infty} f_\perp \propto \exp\left(-\frac{H_{\text{rest}}}{T}\right)$$

$H_{\text{rest}} =$ single particle Hamiltonian of beam in rest frame (energy units)

$T = \text{const}$ Thermodynamic temperature (energy units)

Beam propagation time in transport channel is generally short relative to collision time, inhibiting full relaxation

- Collective effects may enhance relaxation rate
  - Wave spectrums likely large for real beams and enhanced by transient and nonequilibrium effects
  - Random errors acting on system may enhance and lock-in phase mixing

Continuous focusing thermal equilibrium distribution

Analysis of the rest frame transformation shows that the 2D Maxwell-Boltzmann distribution (careful on frame for temperature definition!) is:

$$\boxed{f_\perp(H_\perp) = \frac{m\gamma_b\beta_b^2c^2\hat{n}}{2\pi T}\exp\left(-\frac{m\gamma_b\beta_b^2c^2H_\perp}{T}\right)}$$

$$\begin{aligned} H_\perp &= \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta0}^2\mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2} \\ &= \frac{1}{2}\mathbf{x}_\perp'^2 + \psi \end{aligned}$$

$T = \text{const}$ Temperature (energy units, lab frame)

$n(r=0) = \hat{n} = \text{const}$ on-axis density

$\phi(r=0) = 0$ (reference choice)

The density can then be conveniently calculated in terms of a scaled stream function:

$$n(r) = \int d^2x'_\perp\ f_\perp \quad = \hat{n}e^{-\tilde{\psi}}$$

$$\tilde{\psi}(r) \equiv \frac{m\gamma_b\beta_b^2c^2\psi}{T} = \frac{1}{T}\left(\frac{m\gamma_b\beta_b^2c^2k_{\beta0}^2}{2}r^2 + \frac{q\phi}{\gamma_b^2}\right)$$

and the *x*- and *y*-temperatures are equal and spatially uniform with:

$$\boxed{T_x = \gamma_b m\beta_b^2c^2\frac{\int d^2x'_\perp\ x'^2\ f_\perp}{\int d^2x'_\perp\ f_\perp} = T = \text{const}}$$

Scaled Poisson equation for continuous focusing thermal equilibrium

To describe the thermal equilibrium density profile, the Poisson equation must be solved. In terms of the scaled streamfunction:

$$\boxed{\begin{gathered} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\tilde{\psi}}{\partial\rho}\right) = 1 + \Delta - e^{-\tilde{\psi}} \\ \tilde{\psi}(\rho=0) = 0 \qquad \frac{\partial\tilde{\psi}}{\partial\rho}(\rho=0) = 0 \end{gathered}}$$

Here,

$$\lambda_D = \left(\frac{\epsilon_0 T}{q^2\hat{n}}\right)^{1/2}$$

Debye length formed from the peak, on-axis beam density

$$\rho = \frac{r}{\gamma_b\lambda_D}$$

Scaled radial coordinate in rel. Debye lengths

$$\hat{\omega}_p \equiv \left(\frac{q^2\hat{n}}{\epsilon_0 m}\right)^{1/2}$$

Plasma frequency formed from on-axis beam density

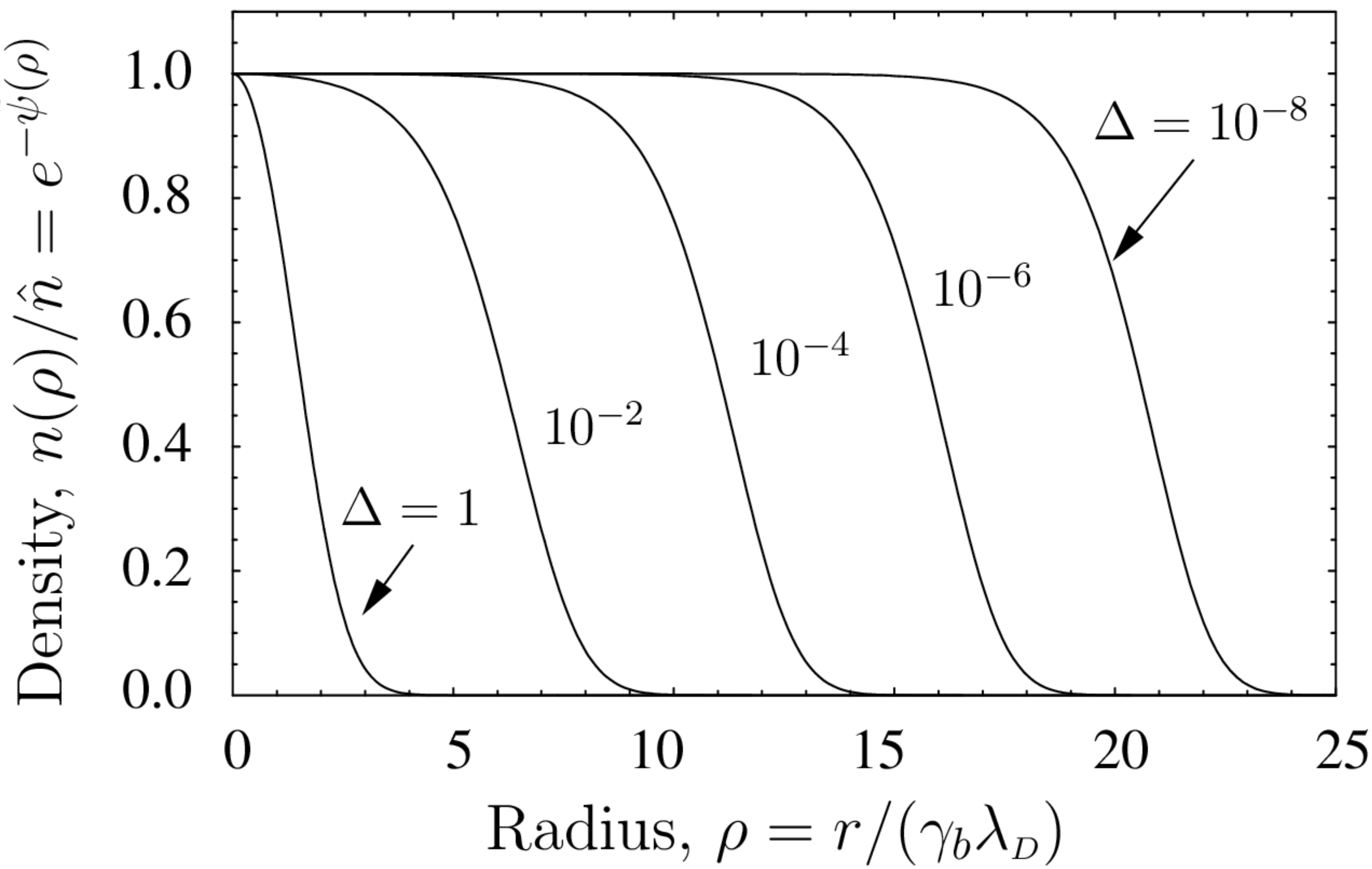
$$\longrightarrow \quad \lambda_D = \left(\frac{T}{\hat{\omega}_p^2 m}\right)^{1/2}$$

$$\Delta = \frac{2\gamma_b^3\beta_b^2c^2k_{\beta 0}^2}{\hat{\omega}_p^2} - 1$$

Dimensionless parameter relating the ratio of applied to space-charge defocusing forces

- Equation is highly nonlinear, but can be solved (approximately) analytically
- Scaled solutions depend only on the single dimensionless parameter Δ

Numerical solution of scaled thermal equilibrium Poisson equation in terms of a normalized density



- Equation is highly nonlinear and must, in general, be solved numerically
  - Dependance on Δ is very sensitive
  - For small Δ, the beam is nearly uniform in the core
- Edge fall-off is always in a few Debye lengths when Δ is small
  - Edge becomes very sharp at fixed beam line-charge

/// Aside: Approximate Analytical Solution for the Thermal Equilibrium Density/Potential

Using the scaled density

$$N \equiv \frac{n}{\hat{n}} = e^{-\tilde{\psi}}$$

the equilibrium Poisson equation can be equivalently expressed as:

$$\boxed{\begin{aligned} &\frac{\partial^2 N}{\partial \rho^2} - \frac{1}{N}\left(\frac{\partial N}{\partial \rho}\right)^2 + \frac{1}{\rho}\frac{\partial N}{\partial \rho} = N^2 - (1+\Delta)N \\ &N(\rho = 0) = 1 \\ &\left.\frac{\partial N}{\partial \rho}\right|_{\rho=0} = 0 \end{aligned}}$$

This equation has been analyzed to construct limiting form analytical solutions for both large and small Δ [see: Startsev and Lund, PoP **15**, 043101 (2008)]

- *Large* Δ solution => warm beam => Gaussian-like radial profile
- *Small* Δ solution => cold beam => Flat core, bell shaped profile
  - Highly nonlinear structure, but approx solution has very high accuracy out to where the density becomes exponentially small!

Large Δ solution:

$$N \simeq \exp\left[-\frac{1+\Delta}{4}\rho^2\right]$$

- Accurate for $\Delta \gtrsim 0.1$ [For full error spec. see: PoP **15**, 043101 (2008)]

Small Δ solution:

$$N \simeq \frac{\left(1+\frac{1}{2}\Delta+\frac{1}{24}\Delta^2\right)^2}{\left\{1+\frac{1}{2}\Delta I_0(\rho)+\frac{1}{24}[\Delta I_0(\rho)]^2\right\}^2}$$

$I_0(x) =$ 0th order Modified Bessel Function of 1st kind

- Highly accurate for $\Delta \lesssim 0.1$ [For full error spec. see: PoP **15**, 043101 (2008)]

Special numerical methods have also been developed to calculate N or $\tilde{\psi} = -\ln N$ to arbitrary accuracy for *any* value of Δ, *however small*
[see: Lund, Kikuchi, and Davidson, PRSTAB, to be published, (2008) Appendices F, G]

- Extreme flatness of solution for small $\Delta \lesssim 10^{-8}$ creates numerical precision problems that require special numerical methods to address
- Method was used to verify accuracy of small Δ solution above

///

Parameters constraints for the thermal equilibrium beam

Parameters employed in $f_\perp(H_\perp)$ to specify the equilibrium are (+ kinematic factors): $\hat{n}, \quad T, \quad \Delta$

Parameters preferred for accelerator applications:

$$k_{\beta 0}, \quad Q, \quad \varepsilon_x = \varepsilon_y = \varepsilon_b$$

Needed constraints can be calculated directly from the equilibrium:

$$\boxed{\begin{aligned} Q &= \left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) \int_0^\infty d\rho \rho \, e^{-\tilde{\psi}} \\ k_{\beta 0}^2 \varepsilon_b &= 4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right)\left[4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) + Q\right] \\ k_{\beta 0}^2 &= \left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) \frac{1+\Delta}{2(\gamma_b \lambda_D)^2} \end{aligned}}$$

Integral function of Δ only

Also useful,

$$\boxed{\begin{aligned} \varepsilon_b^2 &= 16 \frac{T}{\gamma_b m \beta_b^2 c^2} \langle x^2 \rangle_\perp^2 = 4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) r_b^2 \\ r_b^2 &= 4\langle x^2 \rangle_\perp = \frac{1}{k_{\beta 0}^2}\left[4\left(\frac{T}{\gamma_b m \beta_b^2 c^2}\right) + Q\right] \end{aligned}}$$

Example of derivation steps applied to derive previous constraint equations:

Line charge:

$$\lambda = \frac{\gamma_b^2 T}{2q} \int_0^\infty d\rho \; \rho e^{-\tilde{\psi}}$$

rms edge radius:

$$r_b^2 = 4\langle x^2 \rangle_\perp = 2\gamma_b^2 \lambda_D^2 \frac{\int_0^\infty d\rho \; \rho^3 e^{-\tilde{\psi}}}{\int_0^\infty d\rho \; \rho e^{-\tilde{\psi}}}$$

rms edge emittance:

$$\varepsilon_b^2 = \varepsilon_x^3 = 16[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \cancelto{0}{\langle x x' \rangle_\perp^2}]$$

$$= 16 \frac{T}{\gamma_b m \beta_b^2 c^2} \langle x^2 \rangle_\perp = 4 \left(\frac{T}{\gamma_b m \beta_b^2 c^2} \right) r_b^2$$

Matched envelope equation:

$$\cancelto{0}{r_b''} + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_b^2}{r_b^3} = 0$$

These constraints must, in general, be solved numerically

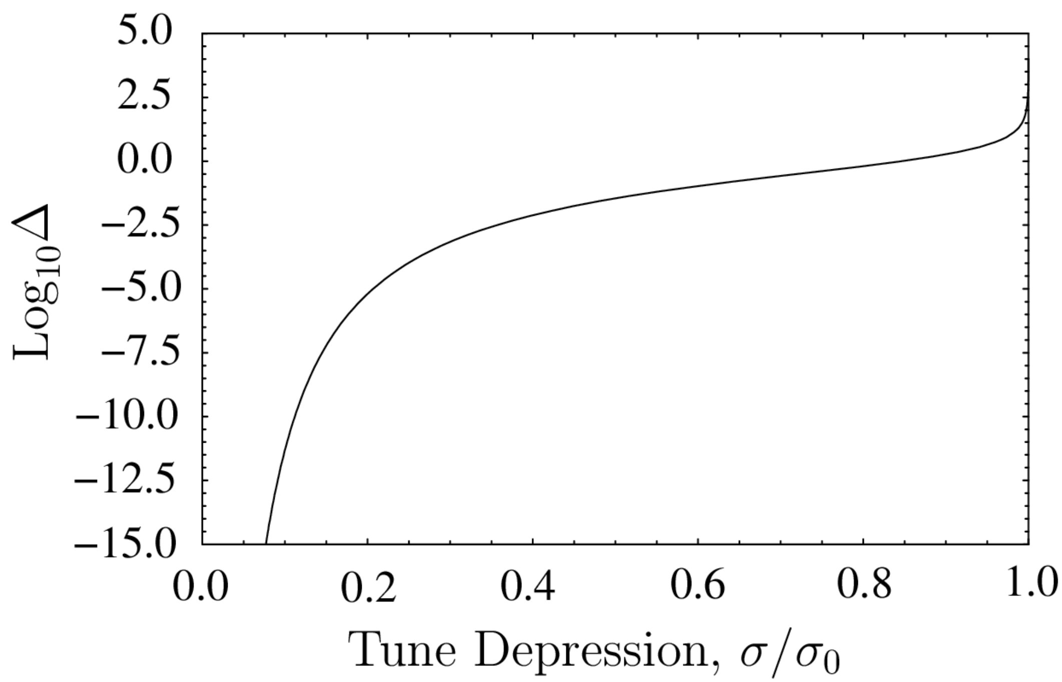
- Useful to probe system sensitivities in relevant parameters

Examples:

1) rms equivalent beam tune depression as a function of Δ

$$\frac{\sigma}{\sigma_0} = \sqrt{1 - \frac{Q}{k_{\beta_0}^2 r_b^2}} = \left\{ 1 - \frac{[\int_0^\infty d\rho\ \rho e^{-\tilde{\psi}}]^2}{(1+\Delta)\int_0^\infty d\rho \rho^3 e^{-\tilde{\psi}}} \right\}^{1/2}$$

R.H.S function of Δ only



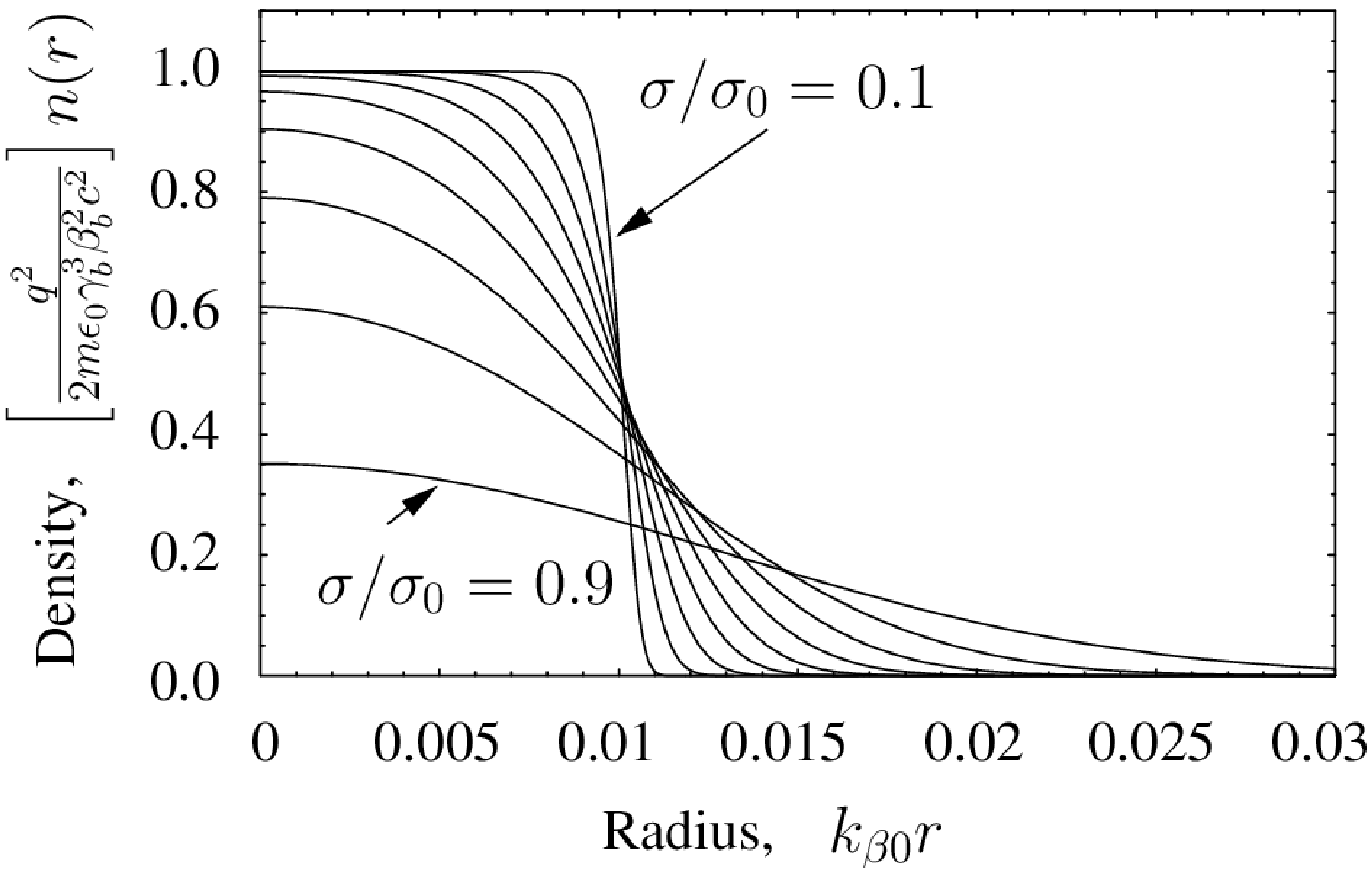
rms equivalent KV measure of σ/σ_0

- Space-charge really nonlinear and the Thermal equilibrium has a spectrum of σ

- Small rms equivalent tune depression corresponds to *extremely* small values of Δ
  - Special numerical methods generally must be employed to calculate equilibrium

2) Density profile at fixed line charge and focusing strength

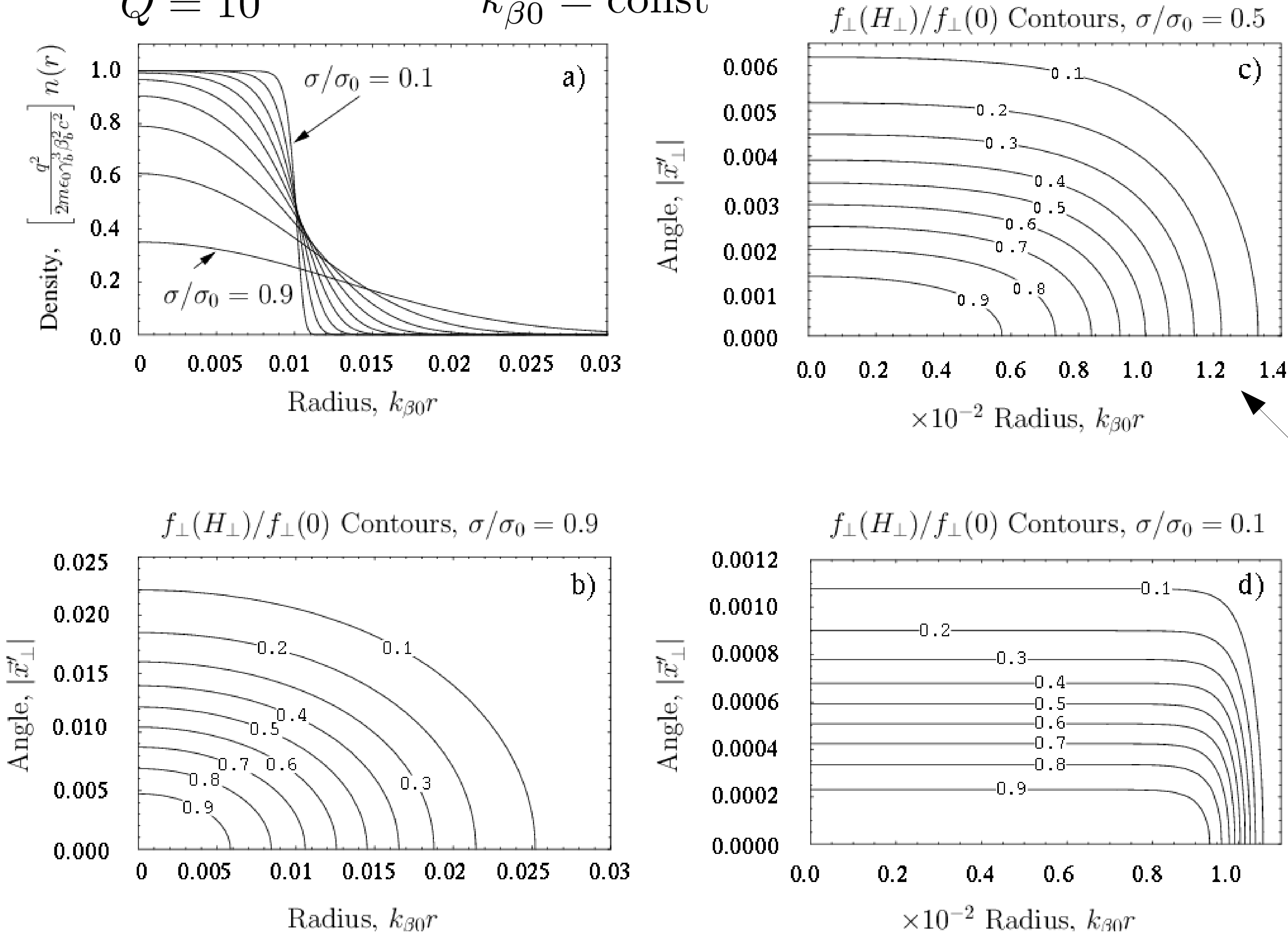
$Q = 10^{-4}$ $\qquad$ $k_{\beta 0}^2 = \text{const}$



- Density profile changes with scaled T
  - Low values yields a flat-top => $\sigma/\sigma_0 \to 0$
  - High values yield a Gaussian like profile => $\sigma/\sigma_0 \to 1$

3) Distribution contours at fixed line charge and focusing strength

$Q = 10^{-4}$ $\qquad$ $k_{\beta 0}^2 = \text{const}$



Radial scales change

- Particles will move approximately force-free till approaching the edge where it is rapidly bent back (see Debye screening analysis this lecture)

Scaled parameters for examples 2) and 3)

			$Q = 10^{-4}$		
σ/σ_0	Δ	s_b	$k_{\beta 0}\gamma_b\lambda_D$	$\frac{T}{m\gamma_b\beta_b^2 c^2}$	$10^3 \times k_{\beta 0}\varepsilon_b$
0.9	1.851	0.3508	12.33	1.065×10^{-4}	0.4737
0.8	6.382×10^{-1}	0.6104	6.034	4.444×10^{-5}	0.2222
0.7	2.649×10^{-1}	0.7906	3.898	2.402×10^{-5}	0.1373
0.6	1.059×10^{-1}	0.9043	2.788	1.406×10^{-5}	0.09375
0.5	3.501×10^{-2}	0.9662	2.077	8.333×10^{-6}	0.06667
0.4	7.684×10^{-3}	0.9924	1.549	4.762×10^{-6}	0.04762
0.3	6.950×10^{-4}	0.9993	1.112	2.473×10^{-6}	0.03297
0.2	6.389×10^{-6}	1.0000	0.7217	1.042×10^{-6}	0.02083
0.1	4.975×10^{-12}	1.0000	0.3553	2.525×10^{-7}	0.01010

Comments on continuous focusing thermal equilibria

From these results it is not surprising that the KV model works well for real beams with strong space-charge (i.e, rms equivalent σ/σ_0 small) since the edges of a smooth thermal distribution become sharp

- Thermal equilibrium likely overestimates the edge with since T = const, whereas a real distribution likely becomes colder near the edge

However, the beam edge contains strong nonlinear terms that will cause deviations from the KV model

- Nonlinear terms can radically change the stability properties (stabilize fictitious higher order KV modes)
- Smooth distributions contain a spectrum of particle oscillation frequencies that are amplitude dependent

S8: Continuous Focusing: Debye Screening in a Thermal Equilibrium Beam

[Davidson, *Physics of Nonneutral Plasmas*, Addison Wesley (1990)]

We will show that space-charge and the applied focusing forces of the lattice conspire together to Debye screen interactions in the core of a beam with high space-charge intensity

- Will systematically derive the Debye length employed by J.J. Barnard in the Introductory Lectures
- The applied focusing forces are analogous to a stationary neutralizing species in a plasma

// Review:

Free-space field of a "bare" test line-charge λ_t at the origin $r = 0$

$$\rho(r) = \lambda_t \frac{\delta(r)}{2\pi r} \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = -\frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

solution (use Gauss' theorem) shows long-range interaction

$$\boxed{\begin{aligned} \phi &= -\frac{\lambda_t}{2\pi\epsilon_0}\ln(r) + \text{const} \\ E_r &= -\frac{\partial \phi}{\partial r} = \frac{\lambda_t}{2\pi\epsilon_0 r} \end{aligned}}$$

//

Place a *small* test line charge at $r = 0$ in a thermal equilibrium beam:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) = -\frac{q}{\epsilon_0}\int d^2x'_\perp \; f_\perp(H_\perp) \; - \; \frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

Thermal Equilibrium   Test Line-Charge

Set:

$$\phi = \phi_0 + \delta\phi$$

$\phi_0 =$ Thermal Equilibrium potential with no test line-charge

$\delta\phi =$ Perturbed potential from test line-charge

Assume thermal equilibrium adapts adiabatically to the test line-charge:

$$n(r) = \int d^2x'_\perp \; f_\perp(H_\perp) \; = \hat{n}e^{-\tilde{\psi}} \; \simeq \hat{n}e^{-\tilde{\psi}_0(r)}e^{-q\delta\phi/(\gamma_b^2 T)}$$

$$\simeq \hat{n}e^{-\tilde{\psi}_0(r)}\left(1 - \frac{q\delta\phi}{\gamma_b^2 T}\right)$$

$$\left|\frac{q\delta\phi}{\gamma_b^2 T}\right| \ll 1$$

Yields:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\delta\phi}{\partial r}\right) = -\frac{q^2}{\epsilon_0\gamma_b^2 T}\hat{n}e^{-\tilde{\psi}_0(r)} \; - \; \frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

Assume a relatively cold beam so the density is flat near the test line-charge:

$$\hat{n}e^{-\tilde{\psi}_0(r)} \simeq \hat{n}$$

This gives:

$$\boxed{\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) - \frac{\delta\phi}{\gamma_b^2\lambda_D^2} = -\frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r} \\ &\lambda_D = \left(\frac{\epsilon_0 T}{q^2\hat{n}}\right)^{1/2} = \text{Debye radius formed from peak, on-axis beam density} \end{aligned}}$$

Derive a general solution by connecting solution very near the test charge with the general solution for r nonzero:

Near solution: $(r \rightarrow 0)$

$$\frac{\delta\phi}{\gamma_b^2\lambda_D^2} \quad \text{Negligible} \dashrightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \delta\phi}{\partial r}\right) = -\frac{\lambda_t}{2\pi\epsilon_0}\frac{\delta(r)}{r}$$

The free-space solution can be immediately applied:

$$\boxed{\begin{aligned} &\delta\phi \simeq -\frac{\lambda_t}{2\pi\epsilon_0}\ln(r) + \text{const} \\ &r \rightarrow 0 \end{aligned}}$$

General Exterior Solution: $(r \neq 0)$

The delta-function term vanishes giving:

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \delta\phi}{\partial \rho}\right) - \delta\phi = 0 \qquad \rho \equiv \frac{r}{\gamma_b \lambda_D}}$$

This is a modified Bessel equation of order 0 with general solution:

$$\delta\phi = C_1 I_0(\rho) + C_2 K_0(\rho)$$

$I_0(x) =$ Modified Bessel Func, 1st kind

$K_0(x) =$ Modified Bessel Func, 2nd kind

$C_1, \quad C_2 =$ constants

Connection and General Solution:

Use limiting forms:

$\rho \ll 1$

$$I_0(\rho) \rightarrow 1 + \Theta(\rho^2)$$

$$K_0(\rho) \rightarrow -[\ln(\rho/2) + 0.5772 \cdots + \Theta(\rho^2)]$$

$\rho \gg 1$

$$I_0(\rho) \rightarrow \frac{e^\rho}{\sqrt{2\pi\rho}}[1 + \Theta(1/\rho)]$$

$$K_0(\rho) \rightarrow \sqrt{\frac{\pi}{2\rho}}[1 + \Theta(1/\rho)]$$

Comparison shows that we must choose for connection to the near solution and regularity at infinity:

$$\boxed{\begin{aligned} C_1 &= 0 \\ C_2 &= \frac{\lambda_t}{2\pi\epsilon_0} \end{aligned}}$$

General solution shows Debye screening of test charge in the core of the beam:

$$\begin{aligned} \delta\phi &= \frac{\lambda_t}{2\pi\epsilon_0} K_0\left(\frac{r}{\gamma_b\lambda_D}\right) \\ &\simeq \frac{\lambda_t}{2\sqrt{2\pi}\epsilon_0}\frac{1}{\sqrt{r/(\gamma_b\lambda_D)}} e^{-r/(\gamma_b\lambda_D)} \qquad r \gg \gamma_b\lambda_D \end{aligned}$$

$K_0(x)$ Order Zero Modified Bessel Function

- Screened interaction does not require overall charge neutrality!
  - Beam particles redistribute to screen bare interaction
  - Beam behaves as a plasma and expect similar collective waves etc.
- Same result for all smooth equilibrium distributions and in 1D, 2D, and 3D
  - Reason why lower dimension models can get the "right" answer for collective interactions in spite of the Coulomb force varying with dimension
- Explains why the radial density profile in the core of space-charge dominated beams are expected to be flat

S9: Continuous Focusing: The Density Inversion Theorem

Shows x and x' dependencies are strongly connected in an equilibrium

For:

$$H_\perp = \frac{1}{2}\mathbf{x}'^2_\perp + \frac{1}{2}k^2_{\beta 0}\mathbf{x}^2_\perp + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

$$f_\perp = f_\perp(H_\perp)$$

$$= \frac{1}{2}\mathbf{x}'^2_\perp + \psi(r)$$

$$\psi \equiv \frac{1}{2}k^2_{\beta 0}r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

calculate the beam density

$$n(r) = \int d^2x'_\perp\ f_\perp(H_\perp) = 2\pi\int_0^\infty dU\ f_\perp(U+\psi(r))$$

differentiate:

$$\frac{\partial n}{\partial \psi} = 2\pi\int_0^\infty dU\ \frac{\partial}{\partial\psi}f_\perp(U+\psi) = 2\pi\int_0^\infty dU\ \frac{\partial}{\partial U}f_\perp(U+\psi)$$

$$= 2\pi\lim_{U\to\infty} \overbrace{f_\perp(U+\psi)}^{0} - 2\pi f_\perp(\psi)$$

bounded distribution

$$\boxed{f_\perp(H_\perp) = -\frac{1}{2\pi}\left.\frac{\partial n}{\partial\psi}\right|_{\psi=H_\perp} \qquad \psi(r) = \frac{1}{2}k^2_{\beta 0}r^2 + \frac{q\phi(r)}{m\gamma_b^3\beta_b^2c^2}}$$

Assume that $n(r)$ is specified, then the Poisson equation can be integrated:

$$\psi(r) - \frac{q\phi(r=0)}{m\gamma_b^3\beta_b^2c^2} = \frac{1}{2}k^2_{\beta 0}r^2 - \frac{q}{m\gamma_b^3\beta_b^2c^2\epsilon_0}\int_0^r\frac{d\tilde{r}}{\tilde{r}}\int_0^{\tilde{r}}d\tilde{\tilde{r}}\ \tilde{\tilde{r}}\ n(\tilde{\tilde{r}})$$

For $n(r)$ = const
$$\int_0^r \frac{d\tilde{r}}{\tilde{r}} \int_0^{\tilde{r}} d\tilde{\tilde{r}}\, \tilde{\tilde{r}}\, n(\tilde{\tilde{r}}) \quad \propto r^2$$

This suggests that $\psi(r)$ is monotonic in r when $d\,n(r)/dr$ is monotonic. Apply the chain rule:

Density Inversion Theorem

$$\boxed{\begin{aligned} f_\perp(H_\perp) &= -\frac{1}{2\pi}\frac{\partial n}{\partial \psi}\bigg|_{\psi=H_\perp} = -\frac{1}{2\pi}\frac{\partial n(r)/\partial r}{\partial \psi(r)/\partial r}\bigg|_{\psi=H_\perp} \\ \psi(r) &= \frac{1}{2}k_{\beta 0}^2 r^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2 c^2} \end{aligned}}$$

For specified monotonic $n(r)$ the density inversion theorem can be applied with the Poisson equation to calculate the corresponding equilibrium $f_\perp(H_\perp)$

Comments on density inversion theorem:

- Shows that the x and x' dependence of the distribution are *inextricably linked* for an equilibrium distribution function $f_\perp(H_\perp)$
  - Not so surprising -- equilibria are highly constrained
- If $df_\perp(H_\perp)/dH_\perp \leq 0$ then the kinetic stability theorem (see: S.M. Lund, lectures on Transverse Kinetic Stability) shows that the equilibrium is also stable

// Example: Application of the inversion theorem to the KV equilibrium

$$n = \begin{cases} \hat{n}, & 0 \leq r < r_b \\ 0, & r_b < r \end{cases} \longrightarrow \frac{\partial n}{\partial r} = -\hat{n}\delta(r - r_b)$$

$$\begin{aligned} \frac{\partial n}{\partial \psi} &= \frac{\partial n/\partial r}{\partial \psi/\partial r} \\ &= -\frac{\hat{n}\delta(r - r_b)}{\partial \psi/\partial r} \\ &= -\frac{\hat{n}\delta(r - r_b)}{\partial \psi/\partial r|_{r=r_b}} \\ &= -\hat{n}\delta(\psi(r) - \psi(r_b)) \end{aligned}$$

property of delta-function:

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|df/dx|_{x=x_i}}$$

$$f(x_i) = 0$$

x_i is root of f

use: $\psi(r_b) = H_\perp|_{\mathbf{x}'_\perp=0} = H_{\perp b}$

$$\longrightarrow \boxed{f_\perp(H_\perp) = -\frac{1}{2\pi}\frac{\partial n}{\partial \psi}\bigg|_{\psi=H_\perp} = \frac{\hat{n}}{2\pi}\delta(H_\perp - H_{\perp b})}$$

Expected KV form //

Similar application of derivatives with respect to Courant-Snyder invariants can "derive" the needed form for the KV distribution of an elliptical beam without guessing.

S10: Comments on the Plausibility of Smooth, Vlasov Equilibria in Periodic Transport Channels

The KV and continuous models are the only (or related to simple transforms thereof) known exact beam equilibria. Both suffer from idealizations that render them inappropriate for use as initial distribution functions for detailed modeling of stability in real accelerator systems:

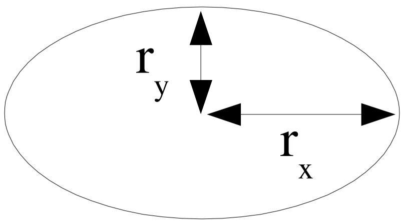
- KV distribution has an unphysical singular structure giving rise to collective instabilities with unphysical manifestations
  - Low order properties (envelope and some features of low-order plasma modes) are physical and very useful in machine design
- Continuous focusing is inadequate to model real accelerator lattices with periodic or *s*-varying focusing forces
  - Kicked oscillator intrinsically different than a continuous oscillator

There is much room for improvement in this area, including study if smooth equilibria exist in periodic focusing and implications if no exact equilibria exist.

Large envelope flutter associated with strong focusing can result in a rapid high-order oscillating force imbalance acting on edge particles of the beam

Temperature Flutter

Elliptical rms Equivalent Beam



$$\varepsilon_x^2 \propto T_x r_x^2 \simeq \text{const} \implies T_x \propto \frac{1}{r_x^2}$$

Example Systems	$(r_{max}/r_{min})^2$
AG Trans: $\sigma_0 = 60^o$	~ 2.5
AG Trans: $\sigma_0 = 100^o$	~ 4.9
Matching Section	~ 15 Possible

Characteristic Plasma Frequency of Collective Effects

Continuous Focusing Estimate

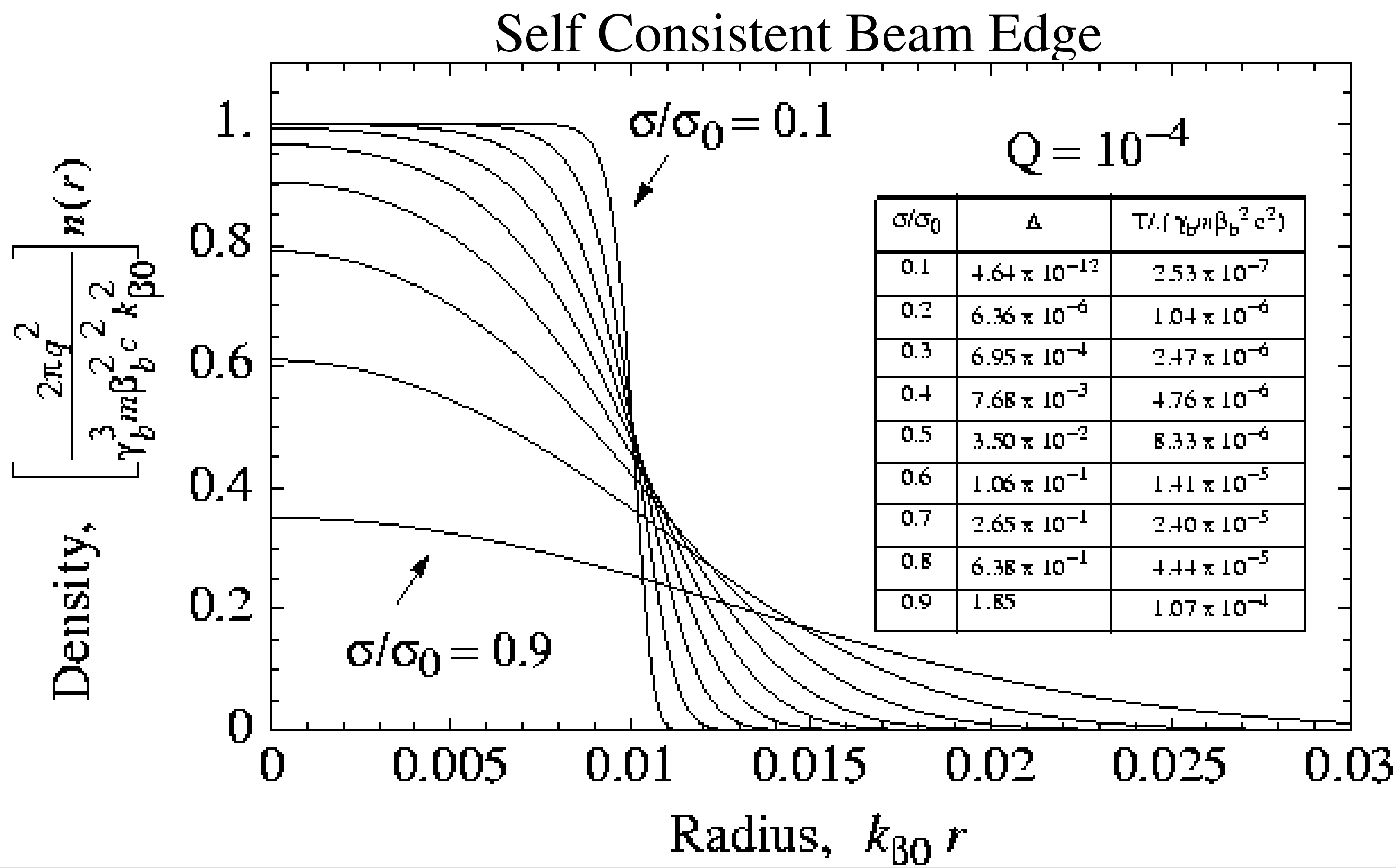
$$\sigma_{\text{plasma}} \sim \frac{L_p}{r_b}\sqrt{2Q} \qquad \text{Typical: } \sigma_{\text{plasma}} \sim 105^\circ/\text{period}$$

- Temperature asymmetry in beam will rapidly fluctuate with lattice periodicity
  - Converging plane => Warmer
  - Diverging plane => Colder
- Collective plasma wave response slower than lattice frequency
  - Beam edge will not be able to adapt rapidly enough
  - Collective waves will be launched from lack of local force balance near the edge

The continuous focusing equilibrium distribution suggests that varying Debye screening together with envelope flutter would require a rapidly adapting beam edge in a smooth, periodic equilibrium beam distribution

$$f_\perp = \frac{m\gamma_b\beta_b^2c^2\hat{n}}{2\pi T}\exp\left(-\frac{m\gamma_b\beta_b^2c^2H_\perp}{T}\right)$$

Continuous Focusing Thermal Equilibrium Beam



σ/σ_0	Δ	$T/(\gamma_b m\beta_b^2 c^2)$
0.1	4.64×10^{-12}	2.53×10^{-7}
0.2	6.36×10^{-6}	1.04×10^{-6}
0.3	6.95×10^{-4}	2.47×10^{-6}
0.4	7.68×10^{-3}	4.76×10^{-6}
0.5	3.50×10^{-2}	8.33×10^{-6}
0.6	1.06×10^{-1}	1.41×10^{-5}
0.7	2.65×10^{-1}	2.40×10^{-5}
0.8	6.38×10^{-1}	4.44×10^{-5}
0.9	1.85	1.07×10^{-4}

It is clear from these considerations that if smooth "equilibrium" beam distributions exist for periodic focusing, then they are highly nontrivial

Would a nonexistence of an equilibrium distribution be a problem:

- Real beams are born off a source that can be simulated
  - Propagation length can be relatively small in linacs
- Transverse confinement can exist without an equilibrium
  - Particles can turn at large enough radii forming an edge
  - Edge can oscillate from lattice period to lattice period without pumping to large excursions

→ Might not preclude long propagation with preserved statistical beam quality

Even approximate equilibria would help sort out complicated processes:

- Reduce transients and fluctuations can help understand processes in simplest form
  - Allows more "plasma physics" type analysis and advances
- Beams in Vlasov simulations are often observed to "settle down" to a fairly regular state after an initial transient evolution
  - Extreme phase mixing leads to an effective relaxation

These notes will be corrected and expanded for reference and future editions of US Particle Accelerator School and University of California at Berkeley courses:

"Beam Physics with Intense Space Charge"

"Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields"

by J.J. Barnard and S.M. Lund

Corrections and suggestions for improvements are welcome. Contact:

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References: For more information see:

M. Reiser, *Theory and Design of Charged Particle Beams*, Wiley (1994, 2008)

R. Davidson, *Theory of Nonneutral Plasmas*, Addison-Wesley (1989)

R. Davidson and H. Qin, Physics of Intense Charged Particle Beams in High Energy Accelerators, World Scientific (2001).

H. Wiedermann, *Particle Accelerator Physics*, Springer-Verlag (1995)

J. Barnard and S. Lund, *Intense Beam Physics*, US Particle Accelerator School Notes, http://uspas.fnal.gov/lect_note.html (2006)

F. Sacherer, *Transverse Space-Charge Effects in Circular Accelerators*, Univ. of California Berkeley, Ph.D Thesis (1968).

S. Lund and B. Bukh, Review Article: *Stability Properties of the Transverse Envelope Equations Describing Intense Beam Transport*, PRST-Accel. and Beams 7, 024801 (2004).

D. Nicholson, *Introduction to Plasma Theory*, Wiley (1983)

I. Kaphinskij and V. Vladimirskij, in *Proc. Of the Int. Conf. On High Energy Accel. and Instrumentation* (CERN Scientific Info. Service, Geneva, 1959) p. 274